

A New Secant-type Method for Solving Nonlinear Equations

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Abstract A new secant-type method for finding zeros of nonlinear equations is presented. In terms of computational cost the new iterative method requires two evaluations of functions per iteration. It is shown and proved that the new method has a convergence of order $1 + \sqrt{2}$. We examine the effectiveness of the new method by approximating the simple root of several nonlinear equations. Numerical examples are given to demonstrate exceptional convergence speed of the proposed method. It is observed that our proposed method is competitive with other similar robust methods and very effective in high precision computations.

Keywords Secant method, Newton method, Simple root, Nonlinear equations, Root-finding, Order of convergence

1. Introduction

Finding the root of nonlinear equations is one of important problem in science and engineering [5]. In this paper, we present a new iterative method to find a simple root α of the nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D is a scalar function. The multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. In recent years, some modifications of the Newton-type methods for simple root have been proposed and analysed [5] and very little work has been done on the secant-type methods. Therefore, the purpose of this paper is to show further development of the secant-type methods. In addition, we will demonstrate the similarity between the Fernandez-Torres and Tiruneh et al. method. We have found that the efficiency index of new iterative method has a better efficiency index than the classical Newton method and equivalent to the Fernandez-Torres, Tiruneh et al. and McDougall et al. methods [1, 3, 6]. Furthermore, we demonstrate the performance of the new method is significantly better when compared with the established methods.

The two well-known iterative methods for finding simple root of nonlinear equations are namely, the classical secant method and the classical Newton method, given by

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] f(x_n), \quad (1)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (2)$$

and their order of convergence is 1.618 and 2 respectively. For the purpose of this paper, we present a new secant-type method for finding simple root of nonlinear equations.

The structure of this paper is as follows: Some basic definitions relevant to the present work are presented in the section 2. In section 3 the new secant-type method and prove the order of convergence are described. In section 4 we demonstrate the equivalency of the two methods recently introduced. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented method.

2. Review of Definitions

In order to establish the order of convergence of an iterative method, following definitions are used [1].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \quad (3)$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant

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[2, 5, 7].

Definition 2 Let $e_k = x_k - \alpha$ be the error in the k th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \tag{4}$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method [2, 5, 7].

Definition 3 Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r, p) = \sqrt[r]{p}, \tag{5}$$

where p is the order of convergence of the method [4].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$COC \approx \frac{\ln |(x_{n+1} - \alpha)(x_n - \alpha)^{-1}|}{\ln |(x_n - \alpha)(x_{n-1} - \alpha)^{-1}|}, \tag{6}$$

further details may be found in [8].

3. The Method and Analysis of Convergence

In this section, we shall define the new iterative method, namely, the secant-type method. To obtain the solution of (1), the new method requires two evaluations of functions and set a particular initial point, ideally close to the simple root. Therefore the expression of the new iterative method for determining the simple root of (1) is given as

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)(x_n - x_{n-1})}{2f'(x_n)^2(x_n - x_{n-1}) - f(x_n)(f'(x_n) - f'(x_{n-1}))} \tag{7}$$

where $n \in \mathbb{N}$, x_0 is the initial value and provided that the denominator of (7) is not equal to zero. It is essential to verify our finding and prove the order of convergence of the new iterative method.

Theorem

Assume that the function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the interval D . If the initial points x_0 and x_1 are sufficiently close to α , then the asymptotic convergence order of the new method defined by (7) is $1 + \sqrt{2}$.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the errors at $(k - 1)$, k and $(k + 1)$

iteration are expressed as $e_{n-1} = x_{n-1} - \alpha$, $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$, respectively.

Using Taylor expansion and taking into account that $f(\alpha) = 0$, we have

$$f(x_{n-1}) = c_1 e_{n-1} + c_2 e_{n-1}^2 + c_3 e_{n-1}^3 + \dots \tag{8}$$

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \dots \tag{9}$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)}, \text{ for } k = 1, 2, 3, \dots \tag{10}$$

Furthermore, we have

$$f'(x_{n-1}) = c_1 + 2c_2 e_{n-1} + 3c_3 e_{n-1}^2 + \dots \tag{11}$$

$$f'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + \dots \tag{12}$$

By substituting the above expressions in the numerator of (7), we obtain

$$2f(x_n)f'(x_n)(x_n - x_{n-1}) - 2c_1^2 e_{n-1} e_n + 2(c_1^2 - 3c_1 c_2 e_{n-1}) e_n^2 + \dots \tag{13}$$

and similarly the denominator of (7) is given as

$$2f'(x_n)^2(x_n - x_{n-1}) - f(x_n)(f'(x_n) - f'(x_{n-1})) = -2c_1^2 e_{n-1} + (2c_1^2 - 6c_1 c_2 e_{n-1} + 3c_1 c_3 e_{n-1}^2 + 4c_1 c_4 e_{n-1}^3) e_n + \dots \tag{14}$$

Dividing (14) by (13) and simplifying, we have

$$e_{n+1} = -\left(\frac{3c_3}{2c_1}\right) e_{n-1} e_n^2 - \left(\frac{2c_4}{c_1}\right) e_{n-1}^2 e_n^2 + \dots \tag{15}$$

In order to prove the order of convergence of (15), we define positive real terms of T_n and T_{n-1} as

$$T_n = \frac{|e_{n+1}|}{|e_n^m|}, \quad T_{n-1} = \frac{|e_n|}{|e_{n-1}^m|}, \tag{16}$$

$$|e_{n+1}| = (T_n T_{n-1}^m) |e_{n-1}^{m^2}|. \tag{17}$$

Therefore, from (15) we obtain

$$\frac{|e_{n+1}|}{|e_n^2| |e_{n-1}|} = \left| \frac{3c_3}{2c_1} \right| = (T_n T_{n-1}^{m-2}) |e_{n-1}^{m^2 - 2m - 1}|. \tag{18}$$

In order to satisfy the asymptotic equation (18), the power of the error term shall approach zero, that is

$$m^2 - 2m - 1 = 0. \tag{19}$$

It is obvious that quadratic equation (19) has two roots,

$$m = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}. \quad (20)$$

The order of convergence of the method (7) is determined by the positive root of (20). Hence, the new method defined by (7) has a convergence order of $1 + \sqrt{2}$. This completes the proof.

The new method (7) requires 2 function evaluations and has the order of convergence $1 + \sqrt{2}$ and to determine the efficiency index of the new method, definition 3 shall be used. Hence, the efficiency index of the new iterative method given by (7) is

$$EI(1 + \sqrt{2}, 2) = \sqrt{2} \approx 1.5538, \quad (21)$$

and the efficiency index of the Newton method and the secant method is given by

$$EI(2, 2) = \sqrt{2} \approx 1.4142, \quad (22)$$

$$EI(2^{-1}(1 + \sqrt{5}), 1) \approx 1.6180, \quad (23)$$

respectively. This indicates that the new method has a better efficiency index than the Newton method and is equivalent to the Fernandez-Torres, Tiruneh et al. and McDougall et al. methods.

4. The Established Methods

For the purpose of comparison, three well-known iterative methods are considered namely, the McDougall et al., the Fernandez-Torres method and the Tiruneh et al. methods. Since these methods are well established, the essential formulas are used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new method. First we will state the Fernandez-Torres method and the Tiruneh et al. method and then demonstrate the similarity between these methods.

In [1], Fernandez-Torres developed a super-quadratic convergence order method for finding simple root of nonlinear equations, the essential expression used in the method is given as,

$$x_{n+1} = x_{n-1} - \frac{f(x_n)(f(x_n) - f(x_{n-1}))(x_{n-1} - x_n)}{f(x_n)(f(x_n) - f(x_{n-1})) - f(x_{n-1})f'(x_n)(x_n - x_{n-1})} \quad (24)$$

The Tiruneh et al. method given in [7] is also a super-quadratic convergence order method for finding simple root of nonlinear equations and is expressed as

$$x_{n+1} = x_{n-1} - (x_{n-1} - x_n)f(x_{n-1}) \left[f(x_{n-1}) - (f(x_n)f(x_{n-1})^{-1}) \right. \\ \left. \times \left[(f(x_n) - f(x_{n-1})) / (x_n - x_{n-1}) \right] \right]^{-1} \quad (25)$$

To demonstrate the equivalency of these methods we will first denote the following

$$\left. \begin{aligned} \Delta x_{n-1} &= x_n - x_{n-1} \\ \Delta f(x_{n-1}) &= f(x_n) - f(x_{n-1}) \end{aligned} \right\} \quad (26)$$

Using (26) and simplifying the Fernandez-Torres method (24), we get

$$x_{n+1} = x_n - \frac{f(x_n)\Delta f(x_{n-1})\Delta x_{n-1}}{f(x_n)\Delta f(x_{n-1}) - f(x_{n-1})f'(x_n)\Delta x_{n-1}} \quad (27)$$

and similarly Tiruneh et al. method (25) is given as

$$x_{n+1} = x_n - \Delta x_{n-1} + \Delta x_{n-1} \left[1 - (f(x_n)f(x_{n-1})^{-1}) \right. \\ \left. \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right] f'(x_n)^{-1} \right]^{-1} \quad (28)$$

To demonstrate the Fernandez-Torres and Tiruneh et al. methods produce identical approximation of the simple root of the nonlinear equation, hence, we simply equate the two methods.

Let (27) = (28), we have

$$-\Delta x_{n-1} + \frac{f'(x_n)\Delta x_{n-1}}{f'(x_n) - (f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f'(x_{n-1})) / \Delta x_{n-1} \right]} \\ = - \frac{\Delta x_{n-1}}{1 - f'(x_n) \left(f(x_{n-1})f(x_n)^{-1} \right) \left[\Delta x_{n-1} / \Delta f(x_{n-1}) \right]} \quad (29)$$

Cancelling common factors, we obtain

$$\frac{f'(x_n)}{f'(x_n) - (f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right]}^{-1} \\ = - \frac{1}{1 - f'(x_n) \left(f(x_{n-1})f(x_n)^{-1} \right) \left[\Delta x_{n-1} / \Delta f(x_{n-1}) \right]} \quad (30)$$

Simplifying, we get

$$\frac{f'(x_n) - \left[f'(x_n) - (f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right] \right]}{f'(x_n) - (f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right]} \\ = - \frac{1}{1 - f'(x_n) \left(f(x_{n-1})f(x_n)^{-1} \right) \left[\Delta x_{n-1} / \Delta f(x_{n-1}) \right]} \quad (31)$$

Further simplification yields

$$\frac{(f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right]}{f'(x_n) - (f(x_n)f(x_{n-1})^{-1}) \left[(\Delta f(x_{n-1})) / \Delta x_{n-1} \right]} \\ = - \frac{1}{1 - f'(x_n) \left(f(x_{n-1})f(x_n)^{-1} \right) \left[\Delta x_{n-1} / \Delta f(x_{n-1}) \right]} \quad (32)$$

Applying cross-multiplication rule to (32), we obtain

$$\left(\frac{f(x_n)}{f(x_{n-1})}\right)\left(\frac{\Delta f(x_{n-1})}{\Delta x_{n-1}}\right)\left[1 - f'(x_n)\left(\frac{f(x_{n-1})}{f(x_n)}\right)\left(\frac{\Delta x_{n-1}}{\Delta f(x_{n-1})}\right)\right] = -\left[f'(x_n) - \left(\frac{f(x_n)}{f(x_{n-1})}\right)\left(\frac{\Delta f(x_{n-1})}{\Delta x_{n-1}}\right)\right] \tag{33}$$

Expanding (33), we have

$$\left(\frac{f(x_n)}{f(x_{n-1})}\right)\left(\frac{\Delta f(x_{n-1})}{\Delta x_{n-1}}\right) - f'(x_n) = -f'(x_n) + \left(\frac{f(x_n)}{f(x_{n-1})}\right)\left(\frac{\Delta f(x_{n-1})}{\Delta x_{n-1}}\right) \tag{34}$$

The equation (34) establishes that the Fernandez-Torres and Tiruneh et al. methods produce identical estimates of the simple root of the nonlinear equation. Furthermore, this is manifested in the numerical examples.

Another super-quadratic convergence order method for finding simple root of nonlinear equations was presented by McDougall et al. [3], we state the essential expressions used in the method.

The initial points are given as

$$x_0^* = x_0, \tag{35}$$

$$\left. \begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{1}{2}[x_0 + x_0^*]\right)} \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}. \end{aligned} \right\} \tag{36}$$

Then the calculation of the simple root of the nonlinear equation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_n + x_n^*]\right)}, \tag{37}$$

where

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{1}{2}[x_{n-1} + x_{n-1}^*]\right)}. \tag{38}$$

5. Numerical Examples

The proposed iterative method given by (7) is employed to solve nonlinear equation with simple root. The difference between the simple root α and the approximation x_n for test function with initial guess x_0 is displayed in tables. Furthermore, the computational order of convergence

approximations are displayed in tables and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value. The results in the tables were obtained by the following algorithm

Algorithm (Maple program)

```
sroots:=fsolve(f(x),x);
root1:=sroots[1];
diff(f(x),x);
x[0]:=x0;
dfx[0]:=diff(f(x),x);
x[1]:=x[0]-fx[0]/dfx[0];
for n from 1 to 6 do
fx[n]:=f(x);
dfx[n]:=diff(f(x),x);
x1[n]:=x[n]-x[n-1];
dfxx[n]:=dfx[n]-dfx[n-1];
x[n+1]:=evalf(x[n]-(2fx[n]dfx[n]x1[n])/(
(2dfx[n]^2x1[n]-fx[n]dfxx[n]));
new[n]:=root1-x[n];
coc:=evalf(ln(new[n]/new[n-1])/(ln(new[n-1]/new[n-2])));
print(n,new[n],coc);
end;
```

Numerical example 1

We will demonstrate the convergence of the new iterative method for the following nonlinear equation

$$f(x) = (x - 2)(x^{10} + x + 1)e^{-x-1}, \tag{39}$$

Having the exact value of the simple root of (39) is $\alpha = 2$. In Table 1 are the errors obtained by the methods described, based on the initial point (IP) $x_0 = 2.002$. We observe that the new iterative method is converging to the expected order.

Numerical example 2

We will demonstrate the convergence of the new iterative method for the following nonlinear equation

$$f(x) = e^x \sin(x) + \ln(1 + x^2), \tag{40}$$

Having the exact value of the simple root of (40) is $\alpha = 0$. In Table 2 are the errors obtained by the methods described, based on the initial value $x_0 = 0.2$.

Numerical example 3

We will demonstrate the convergence of the new method for the following nonlinear equation

$$f(x) = x^6 - x^4 - x^3 - 1, \tag{41}$$

having the exact value of the simple root of (41) is $\alpha = 1.403602$. In Table 3 are the errors obtained by the methods described, based on the initial value $x_0 = 1.25$.

Table 1. Errors occurring in the estimates of the simple root of (39) by the methods described

Method	Second-IP	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	<i>COC</i>
(1)	$x_1 = 2.001$	0.792e-5	0.315e-7	0.994e-12	0.125e-18	1.6522
(2)	$x_1 = 2.001$	0.628e-10	0.157e-19	0.983e-39	0.385e-77	2.0000
(37)	$x_1 = 2.000004$	0.182e-7	0.303e-17	0.153e-41	0.647e-100	2.4162
(24)	$x_1 = 2.000004$	0.497e-12	0.157e-28	0.194e-68	0.937e-165	2.4143
(7)	$x_1 = 2.000004$	0.158e-12	0.100e-29	0.159e-71	0.256e-172	2.4147

Table 2. Errors occurring in the estimates of the simple root of (40) by the methods described

Method	Second-IP	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	<i>COC</i>
(1)	$x_1 = 0.1$	0.256e-1	0.415e-2	0.201e-3	0.165e-5	1.6289
(2)	$x_1 = 0.1$	0.146e-1	0.403e-3	0.324e-6	0.211e-12	2.0000
(37)	$x_1 = 0.01457$	0.402e-2	0.501e-5	0.367e-12	0.247e-29	2.4155
(24)	$x_1 = 0.01457$	0.110e-3	0.865e-9	0.342e-21	0.405e-51	2.4144
(7)	$x_1 = 0.01457$	0.420e-5	0.124e-12	0.324e-31	0.654e-76	2.4135

Table 3. Errors occurring in the estimates of the simple root of (41) by the methods described

Method	Second-IP	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ x_5 - \alpha $	<i>COC</i>
(1)	$x_1 = 1.35$	0.320e-1	0.473e-2	0.390e-3	0.499e-5	1.3782
(2)	$x_1 = 1.35$	0.922e-2	0.222e-3	0.132e-6	0.469e-13	2.0000
(38)	$x_1 = 1.4128$	0.242e-2	0.137e-5	0.188e-13	0.199e-32	2.4144
(24)	$x_1 = 1.4128$	0.892e-4	0.325e-9	0.692e-22	0.112e-52	2.3875
(7)	$x_1 = 1.4128$	0.260e-4	0.291e-10	0.103e-24	0.143e-59	2.4109

6. Conclusions

A new secant-type method for solving nonlinear equations with simple root has been presented. The effectiveness of the new method is examined by showing the accuracy of the simple root of several nonlinear equations. We have shown numerically and verified that the new iterative method has convergence of order 2.414. The major advantage of the new method is that it is very effective and produces high precision of approximation of the simple root. Finally, we conclude that the new method may be considered a very good alternative to the classical methods.

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