A New Family of Optimal Eighth-order Methods for Solving Nonlinear Equations

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Abstract We propose a new family of optimal eight-order methods for solving nonlinear equations. The order of convergence of proposed methods verified using sufficient convergence conditions given in [6]. Using of sufficient convergence condition allows us to develop new optimal three-point iterations. Various numerical examples are considered to check the performance and to verify the theoretical results. Numerical comparisons of proposed methods with some existing methods are made. The test results are in good accordance with our study.

Keywords Nonlinear equations, Iterations methods, Optimal order of convergence

1. Introduction

At present, there are many optimal eighth-order methods for solving nonlinear equations (see, e.g., [1, 2, 3]). They require complicated convergence analysis that is feasible only by symbolic computation, although they produce high accuracy. The interest for these methods has renewed in recent years due to the rapid development of digital computers, advanced computer arithmetic and symbolic computation. In this note, we develop a family of three-point methods with optimal eighth-order convergence. In Section 2, the new family is developed and its convergence analysis is discussed. Unlike the usually considered convergence analysis here we first time used the sufficient conditions under which the three-point iteration have the eighth order of convergence [7, 8]. This allows to simplify the proof of theorem and to reduce tedious calculations. We also discussed similar theorems given by Sharma and Arora in [4, 5] and by Petrovic *et al* in [3]. The theoretical results proved in Section 2 are verified in Section 3 by considering various numerical examples. A comparison of the new methods with the existing methods is also given in this section.

2. The Family of Methods

Let x^* be a simple zero of the function $f(x): D \subset R \rightarrow R$ and x_0 be an initial approximation to x^*. We consider the following simple three-point iteration

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= \phi_4(x_n, y_n), \\
x_{n+1} &= z_n - \alpha_n \frac{f(z_n)}{f'(x_n)}, \quad n = 0, 1, \ldots
\end{aligned}
\tag{1}
$$

Here $\phi_4(x_n, y_n)$ is any two–point optimal fourth order scheme and α_n is given by the following formula

$$
\alpha_n = \frac{f'(x_n)}{(2f[z_n, y_n] - f[z_n, x_n])} \left(\frac{f'(x_n) - f[y_n, x_n]}{f'(x_n) + f[z_n, x_n]} + \tau_n \frac{f[z_n, x_n]}{f'(x_n)} - 1 \right),
\tag{2}
$$

where

$$
f[r, s] = \frac{f(s) - f(r)}{s - r}.
\tag{3}
$$

Note that any two–point optimal fourth order iteration can be written as [8]:

$$
z_n = y_n - \bar{\tau}_n \frac{f(y_n)}{f'(x_n)},
\tag{4}
$$

or

$$
z_n = x_n - \tau_n \frac{f(x_n)}{f'(x_n)},
\tag{5}
$$

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where

$$\bar{\tau}_n = \frac{\tau_n - 1}{\theta_n}, \quad \theta_n = \frac{f(y_n)}{f(x_n)}, \tag{6}$$

$$\tau_n = 1 + \theta_n + 2\theta_n^2 + \beta\theta_n^3 + \gamma\theta_n^4 + \dots,$$

with some constants β and γ.

Theorem 1 *Let the function $f(x)$ be sufficiently differentiable in a neighborhood of its simple zero x^* and $\phi_n(x_n, y_n)$ is an optimal fourth order method. If the initial approximation x_0 is sufficiently close to x^*, then the order of convergence of iteration (1), (2) is 8.*

Proof Using (1), (6) we have

$$f[x_n, y_n] = (1 - \theta_n)f'(x_n),$$

$$f[y_n, z_n] = \frac{1}{\bar{\tau}_n} f'(x_n)(1 - \upsilon_n), \tag{7}$$

$$f[x_n, z_n] = \frac{1}{\tau_n} f'(x_n)(1 - \theta_n\upsilon_n),$$

where

$$\upsilon_n = \frac{f(z_n)}{f(y_n)}. \tag{8}$$

Substituting (7) and (8) into (2) we get

$$\alpha_n = \frac{\tau_n\bar{\tau}_n}{(2\tau_n - \bar{\tau}_n)(1 - \frac{1+\tau_n}{2\tau_n - \bar{\tau}_n}\upsilon_n)}\left(\frac{\tau_n}{\bar{\tau}_n(1 + \frac{\upsilon_n}{\bar{\tau}_n})} - \theta_n\upsilon_n\right), \tag{9}$$

Using the well-known expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \tag{10}$$

One can write (9) as

$$\alpha_n = \frac{\tau_n^2}{2\tau_n - \bar{\tau}_n}(1 + (1 + 2\theta_n)\upsilon_n) + O(\theta_n^4). \tag{11}$$

Since

$$\frac{\tau_n^2}{2\tau_n - \bar{\tau}_n} = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 + \dots,$$

then from (11), we obtain

$$\alpha_n = \left(1 + 2\theta_n + (\beta+1)\theta_n^2 + (2\beta+\gamma-4)\theta_n^3 + \dots\right)\left(1 + (1+2\theta_n)\frac{f(z_n)}{f(y_n)} + O(\theta_n^4)\right) = 1 + 2\theta_n + (\beta+1)\theta_n^2 + (2\beta+\gamma-4)\theta_n^3 +$$

$$+ (1 + 2\theta_n)\frac{f(z_n)}{f(y_n)} + O(\theta_n^4), \tag{12}$$

in which we have used $\frac{f(z_n)}{f(y_n)} = O(\theta_n^2)$. Then by Theorem 1 and 2 in [7, 8] the order of convergence of (1) and (2) is 8.

It is often used two-point iterations (5) with functions [1-5]

$$\tau_n = \frac{1 + (\beta - 1)\theta_n + \beta\theta_n^2}{1 + (\beta - 2)\theta_n}, \tag{13}$$

and

$$\tau_n = \frac{1 - \theta_n + \theta_n^2}{(1 - \theta_n)^2}. \tag{14}$$

In [10] it was developed optimal fourth–order method with parameter

$$\tau_n = \frac{1 - \sqrt{1 - 4\theta_n}}{2\theta_n}. \tag{15}$$

According to (15) we call the iteration (1), (2) with (15) the eighth–order iteration based on Zhanlav's fourth-order method. Similar theorems for (1) presented by Sharma and Arora in [4, 5] under choices

$$\alpha_n = \frac{f'(x)f[z_n, y_n]}{f[z_n, x_n](2f[z_n, y_n] - f[z_n, x_n])}, \tag{16}$$

and

$$\alpha_n = \frac{f'(x) - f[y_n, x_n] + f[z_n, y_n]}{2f[z_n, y_n] - f[z_n, x_n]}, \tag{17}$$

and by Petkovic *et al* in [3] under choice

$$\alpha_n = \frac{f'(x_n)}{2(f[x_n, z_n] - f[x_n, y_n]) + f[y_n, z_n] + (y_n - z_n)f[y_n, x_n, x_n]}, \tag{18}$$

where

$$f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}.$$

Now we consider the three-point iterations (1) with α_n given by formula

$$\alpha_n = \sum_{i=1}^m \omega_i \alpha_n^i, \quad \omega_i \in R, \quad \sum_{i=1}^m \omega_i = 1. \tag{19}$$

Here by α_n^i we denote any functions satisfying the condition (12). In particular, as α_n^i one can take functions (2), (16), (17) and (18). As before, using the sufficient convergence conditions (12) it is easy to prove that the convergence order of three–point iterations (1) with α_n given by (19) is 8.

3. Numerical Experiments

In this section, we have made some numerical experiments on our proposed method and given some numerical comparisons with existing optimal eighth order methods as various examples. We consider the following test functions used in [5, 6]:

$$f_1(x) = \sin^2(x) - x^2 + 1,$$

$$x^* \approx 1.404491648215$$

$$f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$$

$$x^* \approx 1.207647827130$$

$$f_3(x) = \ln(x^2 + x + 2) - x + 1,$$

$$x^* \approx 4.152590736757$$

$$f_4(x) = (x-1)^6 - 1, \qquad x^* = 2$$

All computations have been carried out using Maple 18 computer algebra system with 1500 significant digits and the fixed stopping criterion $\varepsilon = 10^{-250}$. In Tables 1-4, α_n and τ_n with some parameters are considered in first and in second columns, respectively. The number of iterations N, the absolute value $|x_n - x^*|$ and the computational order of convergence (COC) are displayed in theseTables as well. To verify the theoretical order of convergence of our methods, we calculate the computational order of convergence using the formula [6]

$$\rho \approx \frac{\ln(|x_n - x^*|/|x_{n-1} - x^*|)}{\ln(|x_{n-1} - x^*|/|x_{n-2} - x^*|)}.$$

For a comparison, we employed the Sharma-Arora methods with function (16), (17) and method given in [3] with function (18). From Tables 1-4, we see that the COC perfectly coincides with theoretical order and the new method (1), (2) with function (15) is comparable with existing methods.

Table 1: Performance of methods as α_n chosen by (2)

α_n	τ_n		N	$|x^* - x_n|$	COC	N	$|x^* - x_n|$	COC
				$f_1(x)$, $x_0 = 1.2$			$f_2(x)$, $x_0 = -1.3$	
	(15)		3	1.483(-474)	8.00000	3	4.481(-439)	8.00000
	(14)		3	7.146(-422)	8.00000	3	3.594(-453)	8.00000
(2)		$\beta = 2$	3	2.952(-299)	8.00000	3	2.953(-382)	8.00000
	(13)	$\beta = 1$	3	3.907(-402)	8.00000	3	8.460(-417)	8.00000
		$\beta = 0$	3	7.140(-487)	8.00000	3	1.257(-612)	8.00000
				$f_3(x)$, $x_0 = 3.0$			$f_4(x)$, $x_0 = 1.9$	
	(15)		3	3.155(-494)	7.99999	3	1.101(-339)	8.00000
	(14)		3	2.122(-431)	8.00000	3	3.749(-276)	8.00000
(2)		$\beta = 2$	3	2.390(-377)	8.00000	4	1.244(-1326)	8.00000
	(13)	$\beta = 1$	3	2.357(-409)	8.00000	4	2.210(-1921)	8.00000
		$\beta = 0$	3	7.977(-456)	8.00000	3	3.021(-365)	8.00000

Table 2: Performance of methods as α_n chosen by (16)

α_n	τ_n		N	$|x^* - x_n|$	COC	N	$|x^* - x_n|$	COC
				$f_1(x)$, $x_0 = 1.2$			$f_2(x)$, $x_0 = -1.3$	
	(15)		3	7.289(-480)	8.00000	3	1.361(-500)	8.00000
	(14)		3	5.093(-409)	8.00000	3	8.654(-503)	8.00000
(16)		$\beta = 2$	3	6.557(-467)	7.99999	3	5.817(-470)	8.00000
	(13)	$\beta = 1$	3	2.658(-397)	8.00000	3	1.311(-486)	8.00000
		$\beta = 0$	3	1.870(-433)	8.00000	3	2.294(-566)	8.00000
				$f_3(x)$, $x_0 = 3.0$			$f_4(x)$, $x_0 = 1.9$	
	(15)		3	1.369(-566)	7.99999	3	1.177(-380)	8.00000
	(14)		3	4.047(-539)	8.00000	3	8.423(-360)	8.00000
(16)		$\beta = 2$	3	1.754(-527)	8.00000	3	4.742(-262)	8.00000
	(13)	$\beta = 1$	3	6.122(-532)	8.00000	3	7.752(-357)	8.00000
		$\beta = 0$	3	8.825(-550)	8.00000	3	1.013(-386)	8.00000

Table 3: Performance of methods as α_n chosen by (17)

α_n	τ_n		N	$|x^* - x_n|$	COC	N	$|x^* - x_n|$	COC
				$f_1(x)$, $x_0 = 1.2$			$f_2(x)$, $x_0 = -1.3$	
	(15)		3	7.289(-480)	8.00000	3	3.414(-445)	8.00000
	(14)		3	5.093(-408)	8.00000	3	1.170(-437)	8.00000
(17)		$\beta = 2$	3	6.557(-467)	7.99999	3	1.391(-371)	8.00000
	(13)	$\beta = 1$	3	2.658(-396)	8.00000	3	3.217(-406)	8.00000
		$\beta = 0$	3	1.870(-433)	8.00000	3	2.930(-546)	8.00000
				$f_3(x)$, $x_0 = 3.0$			$f_4(x)$, $x_0 = 1.9$	
	(15)		3	6.255(-505)	8.00000	3	4.917(-339)	8.00000
	(14)		3	1.740(-443)	8.00000	4	3.787(-1936)	8.00000
(17)		$\beta = 2$	3	1.017(-390)	8.00000	4	4.192(-1033)	8.00000
	(13)	$\beta = 1$	3	6.576(-422)	8.00000	4	1.241(-1515)	8.00000
		$\beta = 0$	3	2.260(-467)	8.00000	3	1.407(-324)	8.00000

Table 4: Performance of methods as α_n chosen by (18)

α_n	τ_n		N	$|x^* - x_n|$	COC	N	$|x^* - x_n|$	COC
				$f_1(x)$, $x_0 = 1.2$			$f_2(x)$, $x_0 = -1.3$	
	(15)		3	1.583(-466)	8.00000	4	1.543(-476)	8.00000
	(14)		3	1.441(-335)	8.00000	4	2.105(-420)	8.00000
(18)		$\beta = 2$	4	1.617(-1945)	7.99999	3	7.034(-363)	8.00000
	(13)	$\beta = 1$	3	1.978(-293)	8.00000	3	2.930(-393)	8.00000
		$\beta = 0$	3	3.409(-402)	8.00000	3	2.356(-506)	8.00000
				$f_3(x)$, $x_0 = 3.0$			$f_4(x)$, $x_0 = 1.9$	
	(15)		3	7.134(-489)	8.00000	3	1.913(-353)	8.00000
	(14)		3	4.392(-461)	8.00000	3	4.134(-252)	8.00000
(18)		$\beta = 2$	3	6.107(-393)	8.00000	4	4.158(-1133)	8.00000
	(13)	$\beta = 1$	3	5.661(-421)	8.00000	4	3.632(-1597)	8.00000
		$\beta = 0$	4	4.392(-461)	8.00000	3	2.717(-330)	8.00000

4. Conclusions

In this paper, a new family of optimal eight-order methods for solving nonlinear equations is introduced and studied. This family (1), (19) includes the three-point methods given by Sharma and Arora in [4, 5] and by Petkovic *et al* in [3] and our proposed method (1), (2) as particular cases. Finally, the theoretical proofs and numerical experiments have shown that new iterative method is of eight-order and effective.

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