American Journal of Computational and Applied Mathematics 2017, 7(6): 143-148
DOI: 10.5923/j.ajcam.20170706.01

A Numerical Integration for Solving First Order Differential Equations Using Gompertz Function Approach

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Abstract In this paper, we present a new numerical integration of a derived interpolating function using the Gompertz Function approach for solving first order differential equations. The new numerical integration obtained was used to solve some oscillatory and exponential problems. The effectiveness of the new Integrator was verified and the results obtained show that the Integrator is computational reliable and stable.

Keywords Numerical integration, Oscillatory and exponential problems, Gompertz Function, Interpolating Function

1. Introduction

Differential equations are used to model problems in science and engineering that involve the change of some variables with respect to another. Most of these problems required the solution of an initial value problem. Many scholars [1, 3, 5] have approached the problems in various ways by formulating an interpolating function through which an integration scheme which is particularly well suited were developed and used to solve the problems. Similarly, Mathematician scholars have approached the growth problems by formulating functions and distributions which play an important role in modeling survival times, human mortality and actuarial data. [2, 4]

Gompertz proposed and showed that if the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction. The number of survivors was given by the equation

$$L_x = kg^{e^x}. \tag{1}$$

Winsor in 1932 for more convenient was written in the form

$$y = ke^{-e^{a-bx}} \tag{2}$$

in which k and b are essentially positive quantities.

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Published online at http://journal.sapub.org/ajcam

The Gompertz curve or function which has been useful for the empirical representation of growth phenomena. These functions have for long been applied to Actuarial and Demographic problems such as: growth of organisms, psychological growth and population growth.

In this research, we consider an interpolating function in similitude to Gompertz function with additional terms added and derived a numerical integration that can compete favorably in solving some physical problems of growth phenomena, the two parameters (Scale and shape) were considered in the formulation.

Therefore, we formulate or derived new computational numerical integration scheme from the method based on the theoretical solution $y(x)$ to the initial value problem of the form:

$$y^{'} = f(x, y), \quad y(x_0) = y_0 \tag{3}$$

in the interval $[x_n, x_{n+1}]$ by considering the interpolating function

$$\frac{F(x,y)}{K} = \alpha_1 e^{\beta x} + \alpha_2 B^x + \alpha_3 cosx \tag{4}$$

Where $\alpha_1, \alpha_2, \alpha_3$ are real undetermined coefficients, $\beta \; and \; B$ are the shape and scale parameters, K represent the saturation level using Gompertz approach. The intervals defined are $x \in [0,1]$ and $k \in (0,1]$.

Some conditions will be imposed on (4) to guarantee the derivation of the method. The numerical experiments will be carried out on some initial value problems and the test results show the reliability of the method.

2. Derivation of the Integrator

We can assume that the theoretical solution $y(x)$ to the initial value problem (3) can be locally represented in the interval $[x_{n,}\ x_{n+1}\ ], n \geq 0$ by the non-polynomial interpolating function (4) above, that is

$$\frac{F(x,y)}{K} = \alpha_1 e^{\beta x} + \alpha_2 B^x + \alpha_3 cosx$$

We shall assume y_n is a numerical estimate to the theoretical solution $y(x)$ and $f_n = f(x_n, y_n)$.

We define mesh points as follows: $x_n = a + nh,$ and

$$x_{n+1} = a + (n+1)h, n = 0, 1, 2, \ldots \tag{5}$$

We can impose the following constraints on the interpolating function (4) in order to get the undetermined coefficients.

2.1. Constraints

a. The interpolating function must coincide with the theoretical solution at $x = x_n\ and\ x = x_{n+1}$. Hence we required that

$$F(x_n, y_n) = K(\alpha_1 e^{\beta x_n} + \alpha_2 B^{x_n} + \alpha_3 cosx_n) \tag{6}$$

and

$$F(x_{n+1},\ y_{n+1}) = K(\alpha_1 e^{\beta x_{n+1}} + \alpha_2 B^{x_{n+1}} + \alpha_3 cosx_{n+1}) \tag{7}$$

b. Secondly, the derivatives of the interpolating function are required to coincide with the differential equation as well as its first, second, and third derivatives with respect to $x\ at\ x = x_n$

We denote the *i*-th total derivatives of $f(x,y)$ with respect to x with $f^{(i)}$ such that

$$F^1(x_n) = f_n,\ F^2(x_n) = f_n^1, \quad F^3(x_n) = f_n^2 \tag{8}$$

2.2. Derivation of the Numerical Integrator

Following from (8), we differentiate (4) to obtain

$$f_n = k\alpha_1 \beta e^{\beta x_n} + k\alpha_2 B^{x_n}\ log\, B - k\alpha_3 sinx_n \tag{9}$$

$$f_n{}^1 = k\alpha_1 \beta^2 e^{\beta x_n} + k\alpha_2 B^{x_n} (log\, B)^2 - k\alpha_3 cosx_n \tag{10}$$

$$f_n{}^2 = k\alpha_1 \beta^3 e^{\beta x_n} + k\alpha_2 B^{x_n} (log\, B)^3 + k\alpha_3 sinx_n \tag{11}$$

We formed a system of equation to solve for $\alpha_1, \alpha_2, and\ \alpha_3$ from (9) to (11),

Hence we have,

$$\begin{pmatrix} K\beta e^{\beta x_n} & KB^{x_n} logB & -Ksinx_n \\ K\beta^2 e^{\beta x_n} & KB^{x_n}(logB)^2 & -Kcosx_n \\ K\beta^3 e^{\beta x_n} & KB^{x_n}(logB)^3 & Ksinx_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} f_n \\ f_n{}^1 \\ f_n{}^2 \end{pmatrix} \tag{12}$$

Taking this system of equation such as $AX = B$, using Cramer rule it gives

$$|A| = \begin{matrix} K\beta e^{\beta x_n}((logB)^2 sinx_n + (logB)^3 cosx_n) - KlogBe^{\beta x_n}(\beta^2 sinx_n + \beta^3 cosx_n) \\ -Ksinx_n e^{\beta x_n}(\beta^2 (logB)^3 - \beta^3 (logB)^2)) \end{matrix} \tag{13}$$

$$|X_1| = \begin{matrix} f_n (log\, B)^2 sinx_n + (log\, B)^3 cosx_n - (logB)(f_n^1 sinx_n + f_n^2 cosx_n) \\ -sinx_n (f_n^1 (logB)^3 - f_n^2 (logB)^2) \end{matrix} \tag{14}$$

$$|X_2| = \beta(f_n^1 sinx_n + f_n^2 cosx_n) - f_n(\beta^2 sinx_n + \beta^3 cosx_n) - sinx_n(\beta^2 f_n^2 - \beta^3 f_n^1) \tag{15}$$

$$|X_3| = \beta(logB^2 f_n^2 - logB^3 f_n^1) - logB\ (\beta^2 f_n^2 - \beta^3 f_n^1) + f_n\ (\beta^2 logB^3 - \beta^3 logB^2) \tag{16}$$

Therefore, we can deduce that

$$\alpha_1 = \frac{\begin{matrix} f_n (log\ B)^2 sin\, x_n + (log\ B)^3 cos\, x_n - (logB)(f_n^1 sin\, x_n + f_n^2 cos\, x_n) \\ -sin\, x_n (f_n^1 (logB)^3 - f_n^2 (logB)^2) \end{matrix}}{\begin{matrix} K\beta e^{\beta x_n}((logB)^2 sin\, x_n + (logB)^3 cos\, x_n) - KlogB\, e^{\beta x_n}(\beta^2 sin\, x_n + \beta^3 cos\, x_n) \\ -Ksin\, x_n e^{\beta x_n}(\beta^2 (logB)^3 - \beta^3 (logB)^2)) \end{matrix}} \tag{17}$$

$$\alpha_2 = \frac{\beta\left(f_n^1 \sin x_n + f_n^2 \cos x_n\right) - f_n\left(\beta^2 \sin x_n + \beta^3 \cos x_n\right) - \sin x_n\left(\beta^2 f_n^2 - \beta^3 f_n^1\right)}{K\beta B^{x_n}\left((\log B)^2 \sin x_n + (\log B)^3 \cos x_n\right) - KB^{x_n} \log B\left(\beta^2 \sin x_n + \beta^3 \cos x_n\right) - KB^{x_n} \sin x_n\left(\beta^2 (\log B)^3 - \beta^3 (\log B)^2\right))} \tag{18}$$

$$\alpha_3 = \frac{\beta\left(\log B^2 f_n^2 - \log B^3 f_n^1\right) - \log B\left(\beta^2 f_n^2 - \beta^3 f_n^1\right) + f_n\left(\beta^2 \log B^3 - \beta^3 \log B^2\right)}{K\beta B^{x_n}\left((\log B)^2 \sin x_n + (\log B)^3 \cos x_n\right) - KB^{x_n} \log B\left(\beta^2 \sin x_n + \beta^3 \cos x_n\right) - KB^{x_n} \sin x_n\left(\beta^2 (\log B)^3 - \beta^3 (\log B)^2\right))} \tag{19}$$

Since $F(x_{n+1}) = y(x_{n+1})$ and $F(x_n) = y(x_n)$
Implies that $y(x_{n+1}) = y_{n+1}$ and $y(x_n) = y_n$
Then

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \tag{20}$$

and therefore we shall have from putting (6) and (7) into (20),

$$y_{n+1} - y_n = K(\alpha_1 e^{\beta x_{n+1}} + \alpha_2 B^{x_{n+1}} + \alpha_3 cos x_{n+1}) - K(\alpha_1 e^{\beta x_n} + \alpha_2 B^{x_n} + \alpha_3 cos x_n) \tag{21}$$

Simplifying (21) gives

$$y_{n+1} - y_n = K\alpha_1\left[e^{\beta x_{n+1}} - e^{\beta x_n}\right] - K\alpha_2[B^{x_{n+1}} - B^{x_n}] + K\alpha_3[cos x_{n+1} - cos x_n] \tag{22}$$

Recall that $x_n = a + nh, \quad x_{n+1} = a + (n+1)h$ with $n = 0,1,2 \ldots$ (23)

Therefore, by expansion

$$y_{n+1} - y_n = K\alpha_1 e^{\beta x_n}\left(e^{\beta h} - 1\right) - K\alpha_2 B^{x_n}(B^h - 1) + K\alpha_3[cos(x_n + h) - cos x_n] \tag{24}$$

Hence, (24) can be written compactly as

$$y_{n+1} = y_n + P + Q + R$$

Where

$$\left.\begin{aligned} P &= K\alpha_1 e^{\beta x_n}(e^{\beta h} - 1) \\ Q &= -K\alpha_2 B^{x_n}(B^h - 1) \\ R &= K\alpha_3(cos(x_n + h) - cos x_n) \end{aligned}\right\} \tag{25}$$

Substituting for α_1, α_2, and α_3 from (17, 18, 19) in (25), we have

$$y_{n+1} = y_n + P + Q + R \tag{26}$$

where

$$P = \frac{(e^{\beta h} - 1)\left[\left(f_n \log B - f_n^1 - f_n^1 (\log B)^2 + f_n^2 \log B\right)\sin x_n + \left(f_n (\log B)^2 - f_n^2\right)\cos x_n\right]}{\left[(\log B - \beta^2 - \beta^2 (\log B)^2 + \beta^3 \log B)\sin x_n + (\beta(\log B)^2 - \beta^3)\cos x_n\right]}$$

$$Q = \frac{(1 - B^h)\left[\left(f_n^1 - \beta f_n - \beta f_n^2 + \beta^2 f_n^1\right)\sin x_n + \left(f_n^2 - \beta^2 f_n\right)\cos x_n\right]}{\left[(\log B)^2 - \beta \log B - \beta(\log B)^3 + \beta^2 (\log B)^2\right]\sin x_n + \left[(\log B)^3 - \beta^2 \log B\right]\cos x_n}$$

$$R = \frac{(\cos(x_n + h) - \cos x_n)\left[(\beta f_n - f_n^1)(\log B)^2 + (f_n^2 - \beta^2 f_n)\log B + \left(\beta^2 f_n - \beta f_n^2\right)\right]}{(\log B - \beta - \beta(\log B)^2)\sin x_n + ((\log B)^2 - \beta^2)\cos x_n + \beta^2 \log B}$$

is the new numerical schemes for the solution of the first order differential equation.

3. The Implementation of the Integrator

In this paper, however, we limit the numerical integration (26) to first test on some oscillatory and exponential problems of the first order differential equations so as to show the reliability and stability of the integration before applying it to solve the physical problems of growth phenomena.

Example 1

Using the Integrator (26) to solve the initial value problem

$y' = y, y(0) = 1, in\ the\ interval\ 0 \leq x \leq 1$, The analytical solution
$y(x) = e^x$, $h = 0.1$, $B = 2.3211$, $\beta = 1.162$

Table 1

x_n	Numerical Solution	Exact Solution	Absolute Error
0.000	1.00000000000000	1.00000000000000	0.00000000000000
0.100	1.10517099117536	1.10517091807564	0.00000007309972
0.200	1.22140279240542	1.22140275816017	0.00000003424525
0.300	1.34985917214193	1.34985880757600	0.00000036456593
0.400	1.49182567038575	1.49182469764127	0.00000097274448
0.500	1.64874057181637	1.64872127070012	0.00001930111625
0.600	1.82212213083945	1.82211880039050	0.00000333044895
0.700	2.01375800650951	2.01375270747047	0.00000529903904
0.800	2.22554893530880	2.22554092849246	0.00000800681634
0.900	2.45960780466587	2.45960311115695	0.00000469350892
1.000	2.71828853611294	2.71828182845904	0.00000670765390

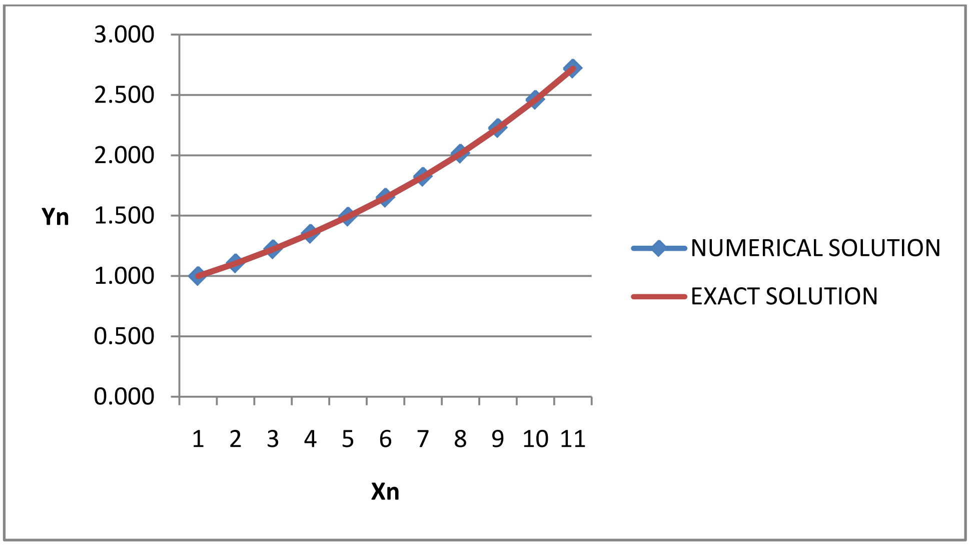

Figure 1. The graph of the Numerical and Exact solution of $y' = y$

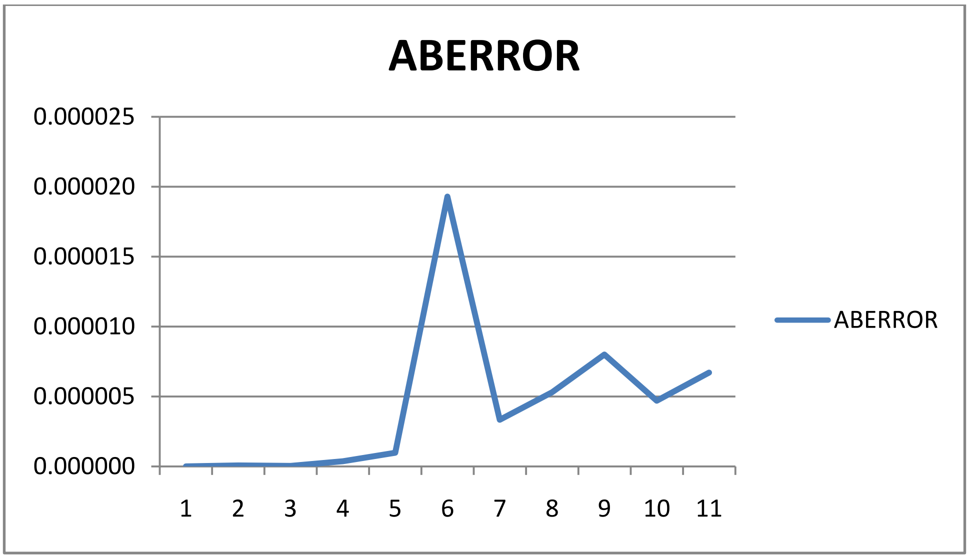

Figure 2. The graph of the Absolute Error of Numerical and Exact solution of $y' = y$

Example 2

Using the Integrator (26) to solve the initial value problem

$y' = \sec(x), y(0) = 1, \mathit{in\ the\ interval}\ 0 \le x \le 1$, The analytical solution $y(x) = 1 + \ln|\sec(x) + \tan(x)|$ $h = 0.025,\ B = 2.3211,\ \beta = 1.740$

Table 2

x_n	Numerical Solution	Exact Solution	Absolute Error
0.000	1.000000000000	1.000000000000	0.000000000000
0.025	1.025003028736	1.025002604574	0.000000424162
0.050	1.050023025525	1.050020846364	0.000002179162
0.750	1.075077768547	1.075070411539	0.000007357008
0.100	1.100181762244	1.100167084547	0.000014677696
0.125	1.125350524847	1.125326798199	0.000023726648
0.150	1.150599787384	1.150565684890	0.000034102494
0.175	1.175945542783	1.175900129383	0.000045413400
0.200	1.201404097137	1.201346823568	0.000057273570
0.225	1.226992123534	1.226922823664	0.000069299870
0.250	1.252726718891	1.252645610358	0.000081108534
0.275	1.278625464325	1.278533152422	0.000092311903
0.300	1.304706489577	1.304603974402	0.000102515176
0.025	1.330988542151	1.330877229036	0.000111313115
0.350	1.357491061847	1.357372775148	0.000118286699
0.375	1.384234261502	1.384111261830	0.000122999672
0.400	1.411239214850	1.411114219869	0.000124994982
0.425	1.438527952541	1.438404161469	0.000123791072
0.450	1.466123567529	1.466004689503	0.000118878026
0.475	1.494050331233	1.493940617682	0.000109713551
0.500	1.522333822088	1.522238103278	0.000095718810

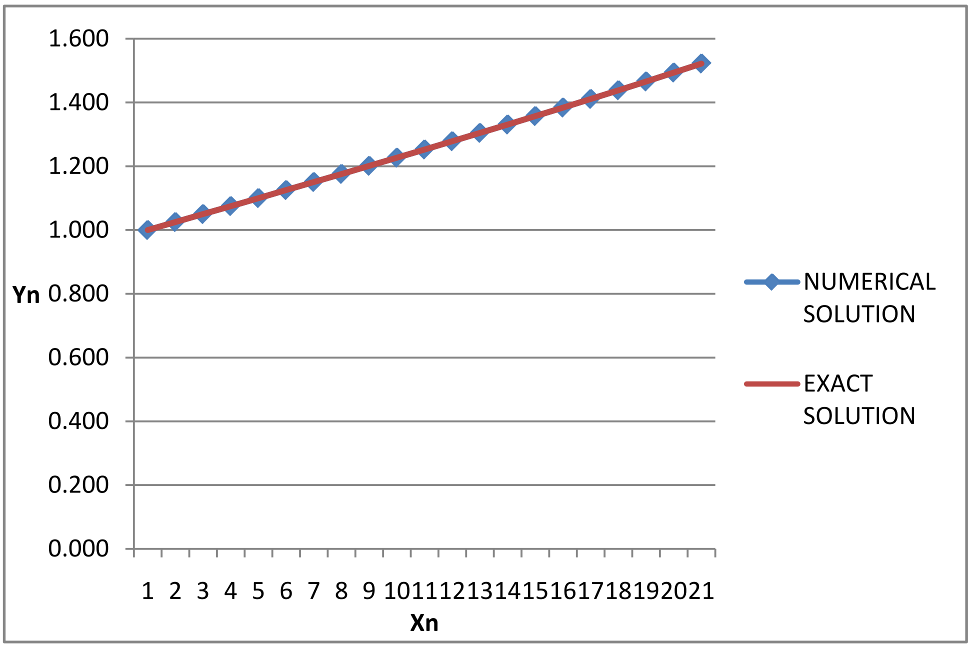

Figure 3. The graph of the Numerical and Exact solution of $y' = \sec(x)$

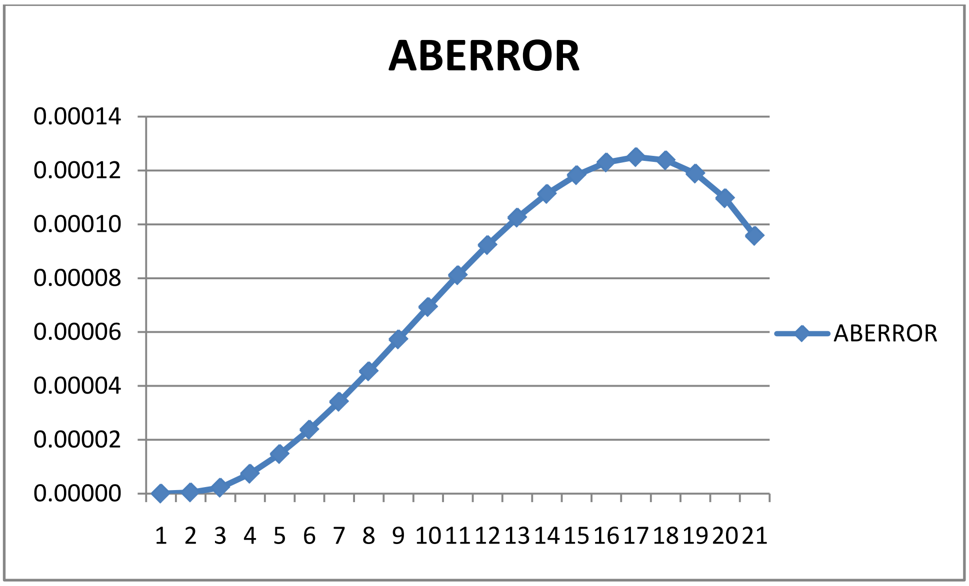

Figure 4. The graph of the Absolute Error of Numerical and Exact solution of $y' = \sec(x)$

4. Conclusions

In conclusion, we have presented a new numerical integration of a derived interpolating function using the Gompertz Function approach for solving first order differential equations. The interpolating function in comparison with Gompertz function with little modification was used, and the scale and the shape parameters were considered. The reliability of the new Integrator was verified and the results obtained show that the Integrator is computational reliable and stable, when the new numerical integration obtained was used to solve some oscillatory and exponential problems.

However the Integrator will be used to solve growth and population problems in the next paper where Gompertz Function and the equation has been widely used.

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