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On Using the Min-Function in Fuzzy Programming within the GAMS Software

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Abstract Using the min-function is essential in some fuzzy programming models. It provides a wider decision space than if it is not used. In some cases, utilizing the min-function in a model within the General Algebraic Modeling System (GAMS) software may not lead to an optimal solution, since this function is not differentiable and the CONOPT solver cannot always find a solution to this type of model. In this paper, the importance of using the min-function in some fuzzy programming models is presented. In addition, the smooth approximation for the min-function can be utilized when the GAMS/CONOPT solver fails to reach the optimal solution of the model. A numerical example that illustrates the correctness of the proposed approach is presented.

Keywords Fuzzy programming, Min-function, GAMS software, CONOPT solver, Smooth approximation

1. Introduction

The main philosophies of decision making in fuzzy environments were established by Bellman and Zadeh [3]. These philosophies have been used as the building blocks of fuzzy linear programming [11]. Zimmermann [10] was the first to propose fuzzy programming for solving single- and multi-objective linear programming problems. Recently, fuzzy programming has been applied to different areas of decision making. For instance, selecting projects using fuzzy linear programming [6], selecting the optimal multi-period portfolio under a fuzzy environment [9] and applying a multi-choice fuzzy linear programming problem to a garment manufacturing company [1]. In addition, using water and land resources for irrigation under uncertainty has been optimized by a multi-objective fuzzy programming method [7].

In many fuzzy programming models, the decision maker seeks to optimize the membership functions, as was done by Chen and Tsai [4] and Aköz and Petrovic [2] in fuzzy goal programming. In most of these cases, the membership functions take the form of a piecewise function.

The main objective of this paper is to show the importance of using the min-function in modeling some fuzzy programming problems. Moreover, since the min-function is not differentiable, its smooth approximation is utilized within the GAMS/CONOPT software.

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Let the linear fuzzy constraints be presented as

$$\sum_{j=1}^{n} a_{ij}\, x_j \lesssim b_i, i = 1, 2, \dots, p, \tag{1}$$

$$\sum_{j=1}^{n} a_{ij}\, x_j \gtrsim b_i, i = p+1, p+2, \dots, m, \tag{2}$$

where $\lesssim$ and $\gtrsim$ mean approximately less than or equal to and approximately greater than or equal to, respectively, x_j, $j = 1, 2, \dots, n$, are non-negative decision variables, a_{ij} is the coefficient of the j^{th} decision variable in the i^{th} fuzzy constraint, while b_i represents the right-hand side of the i^{th} fuzzy constraint. The i^{th} membership function for fuzzy constraints (1) and (2) is presented by (3) and (4), respectively, as follows [4]:

$$\mu_i = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \\ \frac{u_i - \sum_{j=1}^{n} a_{ij}\, x_j}{u_i - b_i} & \text{if } b_i < \sum_{j=1}^{n} a_{ij}\, x_j < u_i, \\ 0 & \text{if } \sum_{j=1}^{n} a_{ij}\, x_j \ge u_i, \end{cases} \tag{3}$$

and

$$\mu_i = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} a_{ij}\, x_j \ge b_i, \\ \frac{\sum_{j=1}^{n} a_{ij}\, x_j - l_i}{b_i - l_i} & \text{if } l_i < \sum_{j=1}^{n} a_{ij}\, x_j < b_i, \\ 0 & \text{if } \sum_{j=1}^{n} a_{ij}\, x_j \le l_i, \end{cases} \tag{4}$$

where u_i and l_i are the i^{th} upper and lower tolerance limits for fuzzy constraints (1) and (2), respectively. Hence, the membership functions (3) and (4) may be modeled as follows:

$$\text{Optimize } f(\mu_1, \mu_2, \dots, \mu_p, \mu_{p+1}, \mu_{p+2} \dots, \mu_m) \tag{5}$$

subject to:

$$\mu_i = \frac{u_i - \sum_{j=1}^{n} a_{ij} x_j}{u_i - b_i}, i = 1, 2, \dots, p,$$
$$\mu_i = \frac{\sum_{j=1}^{n} a_{ij} x_j - l_i}{b_i - l_i}, i = p+1, p+2, \dots, m,$$
$$\phi_s(x_1, x_2, \dots, x_n) \le 0, s = 1, 2, \dots, t, \quad (6)$$
$$0 \le \mu_i \le 1, i = 1, 2, \dots, m,$$
$$x_j \ge 0, j = 1, 2, \dots, n,$$

where $\phi_s(x_1, x_2, \dots, x_n) \le 0$ is the s^{th} crisp constraint. The objective function (5) may take the form of maximizing the sum of the membership functions, which is the simple additive model [4], or the form of maximizing the sum of the weighted membership functions, which is the weighted additive model [8]. In the two cases, it is obvious that the solution may be infeasible or may not be the optimum one since the decision space is limited by setting the domain of the first branch of (3) and (4) as $\sum_{j=1}^{n} a_{ij} x_j = b_i$, i.e., in the form of equality instead of inequality. On the other hand, the membership functions may be modeled as follows:

Objective function (5), subject to:

$$\mu_i \le \frac{u_i - \sum_{j=1}^{n} a_{ij} x_j}{u_i - b_i}, i = 1, 2, \dots, p,$$
$$\mu_i \le \frac{\sum_{j=1}^{n} a_{ij} x_j - l_i}{b_i - l_i}, i = p+1, p+2, \dots, m,$$
$$\phi_s(x_1, x_2, \dots, x_n) \le 0, s = 1, 2, \dots, t, \quad (7)$$
$$0 \le \mu_i \le 1, i = 1, 2, \dots, m,$$
$$x_j \ge 0, j = 1, 2, \dots, n.$$

This formulation overcomes the drawback of model (6), whether in the case of the simple additive or the weighted additive objective function. However, in the case of fuzzy goal programming, extra constraints representing the preemptive importance of the membership functions ($\mu_i \ge \mu_k$, $i \ne k$) might be incorporated within model (7). In this situation, the value of μ_k may not represent the actual achieved degree of the k^{th} fuzzy constraint.

2. The Min-Function and Its Smooth Approximation

In this section, the drawbacks of the above formulations are overcome by utilizing the min-function. Thus, the membership functions (3) and (4) are formulated in model (8) as follows:

Objective function (5), subject to:

$$\mu_i = \min \left\{\frac{u_i - \sum_{j=1}^{n} a_{ij} x_j}{u_i - b_i}, 1\right\}, i = 1, 2, \dots, p,$$
$$\mu_i = \min \left\{\frac{\sum_{j=1}^{n} a_{ij} x_j - l_i}{b_i - l_i}, 1\right\}, i = p+1, p+2, \dots, m,$$
$$\phi_s(x_1, x_2, \dots, x_n) \le 0, s = 1, 2, \dots, t, \quad (8)$$
$$\mu_i, x_j \ge 0, i = 1, 2, \dots, m;\ j = 1, 2, \dots, n.$$

According to this formulation, the decision space is not limited, and at the same time, the values of the membership functions should lie between zero and one inclusive. Moreover, this form can be utilized in the case of fuzzy goal programming with any preemptive priority structure of the membership functions. On the other hand, the GAMS software is used since it is considered a high-level modeling system for mathematical programming and optimization. Hence, in the GAMS software, model (8) takes the form of nonlinear programming with discontinuous derivatives (DNLP). The DNLP is the same as nonlinear programming (NLP) except that non-smooth functions, such as the min-function, appear in the model. The main disadvantage in this case is when the optimal solution cannot be found due to the non-differentiability of these types of functions. To overcome this problem, a smooth approximation for the min-function is utilized. Two alternative smooth approximations are provided for the two-argument min-functions within the GAMS documents [5]. The general form of the two approximations is applied to the i^{th} membership function (3) as follows:

$$\mu_i = \frac{2u_i - \sum_{j=1}^{n} a_{ij} x_j - b_i}{2(u_i - b_i)} - 0.5\sqrt{\left(\frac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{u_i - b_i}\right)^2 + \delta^2}, \quad (9)$$

or

$$\mu_i = \frac{2u_i - \sum_{j=1}^{n} a_{ij} x_j - b_i}{2(u_i - b_i)} - 0.5\sqrt{\left(\frac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{u_i - b_i}\right)^2 + \delta^2} + 0.5\delta, \quad (10)$$

while it is applied to the i^{th} membership function (4) as follows:

$$\mu_i = \frac{\sum_{j=1}^{n} a_{ij} x_j + b_i - 2l_i}{2(b_i - l_i)} - 0.5\sqrt{\left(\frac{\sum_{j=1}^{n} a_{ij} x_j - b_i}{b_i - l_i}\right)^2 + \delta^2}, \quad (11)$$

or

$$\mu_i = \frac{\sum_{j=1}^{n} a_{ij} x_j + b_i - 2l_i}{2(b_i - l_i)} - 0.5\sqrt{\left(\frac{\sum_{j=1}^{n} a_{ij} x_j - b_i}{b_i - l_i}\right)^2 + \delta^2} + 0.5\delta, \quad (12)$$

where δ is an appropriate arbitrary value between 0.01 and 0.0001; and the membership functions (9) to (12) should be non-negative.

The impact of using the min-function in fuzzy programming, as well as the application of its smooth approximation, are illustrated by the following numerical example.

3. Illustrative Example

The approach discussed above is going to be clarified by a numerical example in three different cases. Hence, consider the following three fuzzy constraints:

$2x_1 + 5x_2 + 10x_3 \lesssim 150,$
$4x_1 + 7x_2 + 2x_3 \gtrsim 100,$
$5x_1 + 4x_2 + 6x_3 \gtrsim 120.$

Let $u_1 = 170$, $l_2 = 90$, and $l_3 = 100$. Thus, the membership functions of the three fuzzy constraints are as follows:

$$\mu_1 = \begin{cases} 1 & \text{if } 2x_1 + 5x_2 + 10x_3 \le 150, \\ 8.5 - 0.1x_1 - 0.25x_2 - 0.5x_3 & \text{if } 150 < 2x_1 + 5x_2 + 10x_3 < 170, \\ 0 & \text{if } 2x_1 + 5x_2 + 10x_3 \ge 170, \end{cases}$$

$$\mu_2 = \begin{cases} 1 & \text{if } 4x_1 + 7x_2 + 2x_3 \ge 100, \\ 0.4x_1 + 0.7x_2 + 0.2x_3 - 9 & \text{if } 90 < 4x_1 + 7x_2 + 2x_3 < 100, \\ 0 & \text{if } 4x_1 + 7x_2 + 2x_3 \le 90, \end{cases}$$

$$\mu_3 = \begin{cases} 1 & \text{if } 5x_1 + 4x_2 + 6x_3 \ge 120, \\ 0.25x_1 + 0.2x_2 + 0.3x_3 - 5 & \text{if } 100 < 5x_1 + 4x_2 + 6x_3 < 120, \\ 0 & \text{if } 5x_1 + 4x_2 + 6x_3 \le 100. \end{cases}$$

In the three cases, the objective function is to maximize $\mu_1 + \mu_2 + \mu_3$ (simple additive objective function). Also, the three fuzzy constraints are common in the three cases. One crisp constraint is considered and it is going to be changed from one case to another. The CONOPT solver embedded in GAMS win32 23.8.2 software is used in this example.

The first case: Let the crisp constraint be $3x_1 + 2x_2 + 3x_3 \le 57$, then the solution of model (6) is $x_1 = 2.937$, $x_2 = 9.905$, $x_3 = 9.46$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 0.553$, while the solution of model (8) is $x_1 = 0$, $x_2 = 28.5$, $x_3 = 0$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 0.7$. Even though an optimal solution exists for the two models, the solution of model (8) is better than that of model (6) since the value of the objective function in model (8) is 2.7, while in model (6) it is 2.553.

The second case: Let the crisp constraint be $3x_1 + 4x_2 + 3x_3 \le 67$, then the solution of model (6) is infeasible, while the solution of model (8) is $x_1 = 16.5$, $x_2 = 4.375$, $x_3 = 0$, $\mu_1 = 1$, $\mu_2 = 0.662$, and $\mu_3 = 0$.

The third case: Let the crisp constraint be $3x_1 + 2x_2 + x_3 \le 35$, then the solution of model (6) is $x_1 = 1.316$, $x_2 = 10.877$, $x_3 = 9.298$, $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 0.294$, while for model (8) the optimal solution cannot be found due to the non-differentiability of the min-functions. Hence, the smooth approximations (9) and (11) are utilized, with $\delta = 0.0001$, in this model. It is found that the solution is the same as the solution of model (6).

It is obvious that, according to the structure of the example, model (7) gives the optimal solution. Therefore, it is used as a yardstick for the optimal solution in each case. Accordingly, in the first and the second cases, the solution of model (7) is the same as the solution of model (8), while in the third case, it is the same as the solution of model (8) when the smooth approximation is used.

4. Conclusions

This paper presents an approach for modeling the membership functions in fuzzy programming when the fuzzy constraints are linear. By utilizing the min-function, the actual form of the piecewise membership functions can be effectively presented and considered within different mathematical forms and structures of fuzzy programming models. On the other side, the GAMS/CONOPT software is counted as one of the most efficient solvers. Hence, when the optimal solution of some models cannot be found due to using the min-function, an effective smooth approximation is beneficial to represent the membership functions. Eventually, the fuzzy model takes the form of a nonlinear programming model. The given numerical example illustrates the importance of the proposed methodology.

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