

# Further Acceleration of Thukral Third-Order Method for Determining Multiple Zeros of Nonlinear Equations

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**Abstract** A new fourth-order iterative method for finding zeros of nonlinear equations is introduced. In terms of computational cost the new iterative method requires four evaluations of functions per iteration. It is shown and proved that the new method has a convergence of order four. We examine the effectiveness of the new fourth-order method by approximating the multiple roots of several nonlinear equations. Numerical examples are given to demonstrate exceptional convergence speed of the proposed method.

**Keywords** Newton method, Schroder method, Thukral method, Multiple roots, Nonlinear equations, Root-finding, Order of convergence

## 1. Introduction

In this paper, we present a new fourth-order iterative method to find multiple roots of the nonlinear equation  $f(x) = 0$ , where  $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$  for an open interval where  $I$  is a scalar function. Solving nonlinear equations is one of great practical importance in science and engineering [1, 3, 4, 11]. Hence, many modifications of the Newton-type methods for simple roots have been proposed and analysed [3] but little work has been done on multiple roots. In this paper, we concentrate in the case that  $\alpha$  is a root of multiplicity  $m > 1$  of a nonlinear equation, that is  $f^k(\alpha) = 0$ ,  $k = 0, 1, 2, \dots, m-1$  and  $f^m(\alpha) \neq 0$ . The purpose of this study is to develop a new iterative method for finding multiple roots of nonlinear equations of a higher order than the existing iterative methods [3], and show further development of the Thukral third-order method [7]. Our aim is to improve the order of convergence of the Thukral third-order iterative method [7] and in process we shall compare the performance with the established methods namely, the classical Schroder second-order method [5], the Schroder third-order method [3], the Thukral third-order method [7], the Wu et al fourth-order method [12] and the Li et al fourth-order methods [2]. In addition, the proposed fourth-order method is comparable to the established methods.

The remaining sections of the paper are organized as follows. Some basic definitions relevant to the present work

are presented in section 2. In section 3, we define a new fourth-order iterative method and verify the convergence order. In section 4, well-established methods are stated, which will demonstrate the effectiveness of the new fourth-order iterative method. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented method.

## 2. Review of Definitions

In order to establish the order of convergence of an iterative method the following definitions are used [1, 4, 6-11].

**Definition 1** Let  $f(x)$  be a real-valued function with a root  $\alpha$  and let  $\{x_n\}$  be a sequence of real numbers that converge towards  $\alpha$ . The order of convergence  $p$  is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \quad (1)$$

where  $p \in \mathbb{R}^+$  and  $\zeta$  is the asymptotic error constant [1, 4, 6-11].

**Definition 2** Let  $e_k = x_k - \alpha$  be the error in the  $k$ th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \quad (2)$$

is the error equation. If the error equation exists, then  $p$  is the order of convergence of the iterative method [1, 4, 6-11].

**Definition 3** Let  $r$  be the number of function evaluations of the method. The efficiency of the method is measured by

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the concept of efficiency index and defined as

$$EI(r, p) = \sqrt[p]{p}, \quad (3)$$

where  $p$  is the order of convergence of the method [4].

**Definition 4** Suppose that  $x_{n-1}$ ,  $x_n$  and  $x_{n+1}$  are three successive iterations closer to the root  $\alpha$  of (1). Then the computational order of convergence may be approximated by

$$\text{COC} \approx \frac{\ln|\delta_n \div \delta_{n-1}|}{\ln|\delta_{n-1} \div \delta_{n-2}|}, \quad (4)$$

where  $\delta_i = f(x_i) \div f'(x_i)$ , [7-10].

**Definition 5** Suppose that  $x_{n+1}$  is calculated by the Schroder second-order method [5]

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}, \quad (5)$$

and  $x_{n+1}$  is sufficiently close to the root  $\alpha$ . Then the multiplicity  $m$  may be approximated by

$$\hat{m}_1 \approx \left| \frac{f'(x_n)^2}{f'(x_n)^2 - f(x_n)f''(x_n)} \right|. \quad (6)$$

**Definition 6** Suppose that  $x_{n+1}$  is calculated by the Thukral third-order method [7]

$$x_{n+1} = x_n - \left[ 2 \left( f(x_n)f'(x_n)^2 - f(x_n)^2 f''(x_n) \right) \div \left( 2f'(x_n)^3 - 3f(x_n)f'(x_n)f''(x_n) + f'''(x_n)f(x_n)^2 \right) \right] \quad (7)$$

and  $x_{n+1}$  is sufficiently close to the root  $\alpha$ . Then the multiplicity  $m$  may be approximated by

$$\hat{m}_2 \approx \frac{2 \left( f'(x_n)^3 - f(x_n)f'(x_n)f''(x_n) \right)}{2f'(x_n)^3 - 3f(x_n)f'(x_n)f''(x_n) + f'''(x_n)f(x_n)^2}. \quad (8)$$

### 3. The Method and Analysis of Convergence

To derive the new fourth-order method, we consider the Thukral third-order method which is based on unknown multiplicity [7] and is given by

$$x_{n+1} = x_n - \left[ \frac{2 \left( f'(x_n)^3 - f(x_n)f'(x_n)f''(x_n) \right)}{2f'(x_n)^3 - 3f(x_n)f'(x_n)f''(x_n) + f'''(x_n)f(x_n)^2} \right] \times \left( \frac{f(x_n)}{f'(x_n)} \right). \quad (9)$$

We improve the above method by introducing parameters, thus

$$x_{n+1} = x_n - \left[ \frac{\mu_1 f'(x_n)^3 - \mu_2 f(x_n)f'(x_n)f''(x_n)}{\mu_3 f'(x_n)^3 - \mu_4 f(x_n)f'(x_n)f''(x_n) + \mu_5 f'''(x_n)f(x_n)^2} \right] \times \left( \frac{f(x_n)}{f'(x_n)} \right). \quad (10)$$

The new fourth-order iterative method, given by (10), may also be expressed as

$$x_{n+1} = x_n - \frac{\mu_1 t_1 - \mu_2 t_2}{\mu_3 t_1^2 - \mu_4 t_1 t_2 + \mu_5 t_2 t_3}, \quad (11)$$

where

$$t_1 = \frac{f'(x_n)}{f(x_n)}, \quad t_2 = \frac{f''(x_n)}{f'(x_n)}, \quad t_3 = \frac{f'''(x_n)}{f''(x_n)}, \quad (12)$$

$$\left. \begin{aligned} \mu_1 &= 3m(m+1), & \mu_2 &= 3m^2, \\ \mu_3 &= (2m+1)(m+1), & \mu_4 &= 3m(m+1), \\ \mu_5 &= m^2. \end{aligned} \right\} \quad (13)$$

To verify our finding we will prove the order of convergence of the new fourth-order iterative method.

#### Theorem 1

Let  $f: I \subset \mathbb{R}$  be a function for an open interval  $I \subset \mathbb{R}$ . Let  $f(x_n)$  have a multiple root,  $x = \alpha \in I$  with multiplicity  $m > 1$  and  $x_0$  is the initial guess of the multiple root. Assume that  $f(x_n)$  is a sufficiently differentiable function in  $I$ , then iteration defined by the new scheme (10) has fourth-order convergence.

#### Proof

Let  $\alpha$  be a root of multiplicity  $m$ , that is  $f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0$ , and  $f^{(m)}(\alpha) \neq 0$ . Since  $f(x_n)$  is a sufficiently differentiable function, therefore we expand  $f(\alpha)$  about  $x = \alpha$  by the Taylor series. Also let  $e_n = x_n - \alpha$  and using the Taylor series expansion of  $f(x_n), f'(x_n), f''(x_n), f'''(x_n)$ , about  $\alpha$ , we have

$$f(x_n) = \left( \frac{f^{(m)}(\alpha)}{m!} \right) e_n^m \left[ 1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + \dots \right], \quad (14)$$

$$f'(x_n) = \left( \frac{f^{(m)}(\alpha)}{(m-1)!} \right) e_n^{m-1} \left[ 1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + \dots \right], \quad (15)$$

$$f''(x_n) = \left( \frac{f^{(m)}(\alpha)}{(m-2)!} \right) e_n^{m-2} [1 + C_1 e_n + C_2 e_n^2 + C_3 e_n^3 + \dots], \quad (16)$$

$$f'''(x_n) = \left( \frac{f^{(m)}(\alpha)}{(m-3)!} \right) e_n^{m-3} [1 + D_1 e_n + D_2 e_n^2 + D_3 e_n^3 + \dots], \quad (17)$$

where

$$\left. \begin{aligned} T_k &= \frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}, & A_k &= \frac{(m)!T_k}{(m+k)!}, \\ B_k &= \frac{(m-1)!T_k}{(m+k-1)!}, & C_k &= \frac{(m-2)!T_k}{(m+k-2)!}, \\ D_k &= \frac{(m-3)!T_k}{(m+k-3)!}. \end{aligned} \right\} \quad (18)$$

From (14)-(18), we get

$$\frac{f(x_n)}{f'(x_n)} = \frac{e_n}{m} - \frac{T_1 e_n^2}{m^2(m+1)} + \frac{(T_1^2(m+2) - 2mT_2)e_n^3}{m^3(m+1)(m+2)} + \dots, \quad (19)$$

$$\frac{f'(x_n)}{f(x_n)} = \frac{m}{e_n} + \frac{T_1}{(m+1)} - \frac{(T_1^2(m+2) - (m+1)T_2)e_n}{(m+1)^2(m+2)} + \dots, \quad (20)$$

$$\frac{f''(x_n)}{f'(x_n)} = \frac{m-1}{e_n} + \frac{T_1}{m} - \frac{(T_1^2(m+1) - 2mT_2)e_n}{m^2(m+1)} + \dots, \quad (21)$$

$$\frac{f'''(x_n)}{f''(x_n)} = \frac{m-2}{e_n} + \frac{T_1}{(m-1)} - \frac{(T_1^2 m - 2(m-1)T_2)e_n}{m(m-1)^2} + \dots. \quad (22)$$

Substituting appropriate expressions in (10) and simplifying, we have

$$e_{n+1} = \left( \frac{\kappa_1 T_1^3 - \kappa_2 T_1 T_2 + \kappa_3 T_3}{\kappa_4} \right) e_n^4 + O(e_n^5), \quad (23)$$

where

$$\left. \begin{aligned} \kappa_1 &= (m+2)(m+3)(6m+11)T_1^3, \\ \kappa_2 &= 2m(m+3)(9m+17)T_1 T_2, \\ \kappa_3 &= 18m^2(m+1)T_3, \\ \kappa_4 &= 2m^3(m+1)^2(m+2)(m+3). \end{aligned} \right\} \quad (24)$$

The expression (23) establishes the asymptotic error constant for the fourth-order of convergence for the new iterative method defined by (10). This completes the proof.

The new fourth-order method requires four function evaluations and has the order of convergence four. To determine the efficiency index of the new method, definition 3 will be used. Hence, the efficiency index of the new fourth-order iterative method given by (10) is

$$EI(4,4) = \sqrt[4]{4} \approx 1.4142, \quad (25)$$

and the efficiency index of the Thukral third-order iterative method given by (9) is

$$EI(4,3) = \sqrt[4]{3} \approx 1.3161. \quad (26)$$

This indicates that the new fourth-order method has a better efficiency index than the Thukral third-order iterative method.

## 4. The Established Methods

For the purpose of comparison, five well-known iterative methods are considered namely, the classical Schroder second-order method, the Schroder third-order method, the Thukral third-order method, the Wu et al fourth-order method and the Li et al fourth-order methods. Since these methods are well established, we shall state the essential formulas used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new fourth-order method.

In [5], Schroder developed a second-order method for finding multiple roots of nonlinear equations, since this method is well-established we state the essential expressions used in the method,

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}. \quad (27)$$

The classical Schroder third-order method [3] is obtained and is given as

$$x_{n+1} = x_n - \frac{2mf(x_n)f'(x_n)}{(m+1)f'(x_n)^2 - mf(x_n)f''(x_n)}. \quad (28)$$

In [12], Wu et al. developed a fourth-order method for finding multiple roots of nonlinear equations, since this method is well-established we state the essential expressions used in the method,

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (29)$$

$$x_{n+1} = y_n - m \left( \frac{f(y_n)}{f'(y_n)} \right). \quad (30)$$

The first of fourth-order method presented by Li et al. [2] is expressed as

$$x_{n+1} = x_n - \left( \frac{f(x_n)}{\lambda_1 f'(x_n) + \lambda_2 f'(y_n) + \lambda_3 f'(z_n)} \right) \quad (31)$$

where

$$y_n = x_n - \left( \frac{2m}{(m+2)} \right) \left( \frac{f(x_n)}{f'(x_n)} \right), \quad (32)$$

$$z_n = x_n - 2 \left( \frac{m}{m+2} \right)^m \left( \frac{f(x_n)}{f'(y_n)} \right), \quad (33)$$

$$\left. \begin{aligned} \lambda_1 &= \left( \frac{1}{8} \right) \left[ \frac{m^6 - m^5 - 14m^4 + 12m^3 + 48m^2 - 80m + 32}{m(m^3 + 2m^2 - 8m + 4)} \right] \\ \lambda_2 &= \left( -\frac{m}{16} \right) \left[ \frac{3m^4 - 6m^3 - 20m^2 + 40m - 16}{\left( \frac{m}{m+2} \right)^m (m^3 + 2m^2 - 8m + 4)} \right] \\ \lambda_3 &= \left( \frac{1}{16} \right) \left[ \frac{m^3(m^2 - 4)}{\left( \frac{m}{m+2} \right)^m (m^3 + 2m^2 - 8m + 4)} \right] \end{aligned} \right\} \quad (34)$$

The second of fourth-order method presented by Li et al. [2] is given as

$$x_{n+1} = x_n - \lambda_1 \left( \frac{f(x_n)}{f'(x_n)} \right) - \lambda_2 \left( \frac{f(x_n)}{f'(y_n)} \right) - \lambda_3 \left( \frac{f(x_n)}{f'(z_n)} \right) \quad (35)$$

where

$$y_n = x_n - \left( \frac{2m}{m+2} \right) \left( \frac{f(x_n)}{f'(x_n)} \right), \quad (36)$$

$$z_n = x_n - \left( \frac{2m}{m+2} \right) \left( \frac{f(x_n)}{f'(x_n)} \right) + 2 \left( \frac{m}{m+2} \right)^m \times \left( \frac{f(x_n)}{f'(y_n)} \right), \quad (37)$$

$$\left. \begin{aligned} \lambda_1 &= \left( \frac{m}{8} \right) \left[ \frac{m^4 + 4m^3 - 8m + 48}{m^2 + 2m + 6} \right] \\ \lambda_2 &= \left( \frac{1}{4} \right) \left[ \frac{\left( \frac{m}{m+2} \right)^m m(m^3 + 12m^2 + 36m + 32)}{m^2 + 2m + 6} \right] \\ \lambda_3 &= \left( -\frac{1}{8} \right) \left[ \frac{m^2(m^3 + 6m^2 + 12m + 8)}{m^2 + 2m + 6} \right] \end{aligned} \right\} \quad (38)$$

## 5. Application of the New Fourth-Order Iterative Method

The proposed fourth-order iterative method given by (10) is employed to solve nonlinear equations with multiple roots.

The difference between the multiple root  $\alpha$  and the approximation  $x_n$  for test function with initial guess  $x_0$  is displayed in tables. Furthermore, the computational order of convergence approximations are displayed in tables and we observe that this perfectly coincides with the theoretical result. In addition, the difference between the multiplicity  $m$  and the approximation  $\hat{m}$  is also displayed in tables. The numerical computations listed in the table were performed on an algebraic system called Maple and the errors displayed are of absolute value.

### Numerical example 1

We will demonstrate the convergence of the new fourth-order iterative method for the following nonlinear equation

$$f(x) = [(x-1)^3 - 1]^9, \quad (39)$$

having multiplicity  $m=9$  and the exact value of the multiple root of (39) is  $\alpha=2$ . In Table 1 the errors obtained by the new method described are based on the initial value  $x_0=1.9$ . We observe that the new fourth-order iterative method is converging to the expected order.

### Numerical example 2

We will demonstrate the convergence of the new fourth-order iterative method for the following nonlinear equation

$$f(x) = [e^x + x - 2]^6, \quad (40)$$

having multiplicity  $m=6$  and the exact value of the multiple root of (40) is  $\alpha=0.442854\dots$ . In Table 2 the errors obtained by the new method described are based on the initial value  $x_0=2^{-2}$ . We observe that the new fourth-order iterative method is converging to the expected order.

### Numerical example 3

We will demonstrate the convergence of the new fourth-order method for the following nonlinear equation

$$f(x) = [\sin(x)^2 - x^2 + 1]^{99}, \quad (41)$$

having multiplicity  $m=99$  and the exact value of the multiple root of (41) is  $\alpha=1.404491\dots$ . In Table 3 the errors obtained by the new method described are based on the initial value  $x_0=1.6$ . We observe that the new fourth-order iterative method is converging to the expected order.

**Table 1.** Errors occurring in the estimates of the root of (39) by the methods described

<i>method</i>	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ m - \hat{m}_k $	<i>COC</i>
(27)	0.115e-1	0.131e-3	0.171e-7	0.292e-15	-	2.0000
(28)	0.773e-3	0.308e-9	0.195e-28	0.497e-86	$\hat{m}_1 = 0.233e-6$	3.0000
(30)	0.131e-3	0.292e-15	0.729e-62	0.283e-248	-	4.0000
(31)	0.306e-3	0.171e-13	0.167e-54	0.150e-218	-	4.0000
(35)	0.458e-3	0.126e-12	0.725e-51	0.791e-204	-	4.0000
(10)	0.377e-4	0.673e-18	0.682e-73	0.722e-293	$\hat{m}_2 = 0.455e-32$	4.0000

**Table 2.** Errors occurring in the estimates of the root of (40) by the methods described

<i>method</i>	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ m - \hat{m}_k $	<i>COC</i>
(27)	0.112e-1	0.379e-4	0.438e-9	0.585e-19	0.	2.0000
(28)	0.953e-4	0.761e-14	0.389e-44	0.508e-135	$\hat{m}_1 = 0.950e-9$	2.9960
(30)	0.379e-4	0.585e-19	0.331e-78	0.339e-315	0.	4.0000
(31)	0.459e-5	0.274e-23	0.347e-96	0.898e-388	0.	4.0000
(35)	0.209e-4	0.385e-20	0.444e-83	0.787e-335	0.	4.0000
(10)	0.101e-4	0.863e-22	0.456e-90	0.355e-363	$\hat{m}_2 = 0.179e-31$	4.0000

**Table 3.** Errors occurring in the estimates of the root of (41) by the methods described

<i>method</i>	$ x_1 - \alpha $	$ x_2 - \alpha $	$ x_3 - \alpha $	$ x_4 - \alpha $	$ m - \hat{m}_k $	<i>COC</i>
(27)	0.234e-1	0.415e-3	0.135e-6	0.143e-13	0.	1.9999
(28)	0.307e-2	0.151e-7	0.182e-23	0.319e-71	$\hat{m}_1 = 0.208e-4$	3.0000
(30)	0.415e-3	0.143e-13	0.200e-55	0.777e-223	0.	4.0000
(31)	0.106e-2	0.188e-11	0.187e-46	0.180e-186	0.	4.0000
(35)	0.130e-2	0.540e-11	0.162e-44	0.130e-178	0.	4.0000
(10)	0.262e-3	0.101e-14	0.229e-60	0.598e-243	$\hat{m}_2 = 0.235e-21$	4.0000

## 6. Conclusions

A new fourth-order iterative method for solving nonlinear equations with multiple roots has been introduced. Simply introducing new parameters in the Thukral third-order method, we have achieved a fourth-order of convergence. The effectiveness of the new fourth-order method is examined by showing the accuracy of the multiple roots of several nonlinear equations. In practice if multiplicity  $m$  is unknown then we may use the formula (8) to obtain the approximated value and take the integer part as the multiplicity  $m$ . We have shown numerically and verified that the new iterative method has convergence of order four. The major advantages are the new fourth-order is based on one-point one-step iteration and is simple to compute. Finally, we conclude that the new method may be considered a very good alternative to the classical methods.

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