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Existence and Uniqueness of Nonlinear Implicit Fractional Differential Equation with Riemann-Liouville Derivative

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Abstract We study an initial value problem for nonlinear implicit fractional differential equation with Riemann-Liouville fractional derivative. In the process, we obtain the existence and uniqueness of solutions of an implict fractional differential equation by Banach fixed point theorem. Further, we discuss the uniqueness of solutions via the application of Bihari and Medved inequalities.

Keywords Riemann-Liouville fractional derivative, Fractional integral, Fixed point theorem, Implicit fractional differential equation, Bihari and Medved inequalities

1. Introduction

The study of fractional calculus that grows out of traditional concepts of the calculus derivative and integral operators. Several authors were introduced many different forms of noninteger differential operators and discussed varous results on existence, uniqueness; and qualitative and quantative properties of solutions for fractional differential equations, the reader referred to [7, 11, 12, 16, 23] and the monographs: Samko et al. (1993); Miller et al. (1993); Podlubny (1999); Hilfer (2000); Kilbas et al. (2006); Cresson (2007); Diethem K. (2010); Katugampola (2011) and Abbas S. et al. (2012).

Podlubny I. [20], studied the existence and uniqueness of an initial value problem:

$$D^{\alpha}x(t) = f\big(t, x(t)\big), \tag{1.1}$$

$$D^{\alpha-1}x(t)\big|_{t=0} = x_0 \in \mathbb{R}, \tag{1.2}$$

where $0 < \alpha < 1$, $0 \le t < T \le \infty$, $f: [0, T) \times \mathbb{R} \to \mathbb{R}$; $\mathbb{R}$ denotes the real space and D^{α} denotes Riemann-Liouville fractional derivative operator.

Recently, Chinchane V. L. and Pachpatte D. B. [2] have discussed the uniqueness of solution of fractional differential equation with the Riemann-Liouville derivative. Existence and uniqueness of an implicit fractional differential equations via the Liouville-Caputo derivative have studied by authors in [17] using the fixed point concepts. Kucche et al. [10] investigated existence, uniqueness, continuous dependence and estimates of solutions for an implicit fractional differential equations.

Motivated by the above mentioned works in this manuscript, we discuss the existence and uniqueness of the solution for the following implicit fractional differential equations with Riemann-Liouville derivative:

$$D^{\alpha}x(t) = f\big(t, x(t), D^{\alpha}x(t)\big), \tag{1.3}$$

$$D^{\alpha-1}x(t)\big|_{t=0} = x_0 \in \mathbb{R}, t \in J = [0, b] \tag{1.4}$$

where D^{α} $(0 < \alpha < 1)$ denotes Riemann-Liouville fractional derivative operator and f is real conitnuous valued function on $J \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; $\mathbb{R}$ denotes the real space.

Furthermore, our intention is to extend the results presented by Chinchane V. L. and Pachpatte D. B. to nonlinear implicit fractional differential equation.

The paper is organized as follows. In Section 2, some definitions, lemmas and preliminary results are intoduced to be used in the sequel. Section 3 will involve the assumptions and main result of existence and uniqueness by fixed point theorerm. Finally Section 4 deal the results of uniqueness for the problem (1.3)-(1.4) via inequalities.

2. Preliminaries

Let us recall some definitions and concepts of the fractional calculus [9, 10, 15, 17, 20, 21] and state the few results which are used throuhout this paper.

Definition 2.1. The fractional derivative of order $0 < \alpha < 1$ of a continuous function $f : \mathbb{R}^{+} \to \mathbb{R}$, is given by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} f(s)ds, \tag{2.1}$$

provided that the right side is pointwise defined on $\mathbb{R}^{+}$.

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Definition 2.2. The fractional primitive of order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$, is given as follows

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad (2.2)$$

provided that the right side is pointwise defined on $\mathbb{R}^+$, where $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$.

In [19], Medved introduced a special class of nonlinear functions and developed a method to estimate solution for nonlinear integral inequalities with singular kernel. The functions of such class are defined as follows:

Definition 2.3. Let $q > 0$ be a real number and $0 < b \leq \infty$. The function w: $\mathbb{R}^+ \to \mathbb{R}$ satisfies the following condition

$$e^{-qt}[w(u)]^q \leq R(t)w(e^{-qt}u^q), \qquad (2.3)$$

for all $u \in \mathbb{R}_+, t \in [0,b]$, where $R(t)$ is a continuous, nonnegative function.

Remark 2.1. If $w(u) = u^m, m > 0$, then

$$e^{-qt}[w(u)]^q = e^{(\mathrm{m}-1)qt}w(e^{-qt}u^q), \qquad (2.4)$$

for any $q > 1$. i.e the condition (2.3) is satisfies with $R(t) = e^{(\mathrm{m}-1)qt}$. For $w(u) = u + au^m$, where $0 \leq a \leq 1, m \geq 1$, the function w satisfies the condition (2.3) with $q > 1$ and $R(t) = 2^{q-1}e^{qmt}$, see [14].

Lemma 2.1 [19] Let $0 \leq T \leq \infty$, $u(t), b(t), a(t), a'(t) \in C([0,T), R+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ be nondecreasing function, $w(0) = 0, w(u) > 0$ on $(0,T)$, and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)w\big(u(s)\big)ds, \qquad (2.5)$$

for $t \in [0,T)$ where $\beta > 0$ is constant. Then following hold:

(i) Suppose $\beta > \frac{1}{2}$, and if w satisfies the condition (2.3) with $q = 2$, then

$$u(t) \leq e^t\{\Omega^{-1}[\Omega(2\mathrm{a}(\mathrm{t})^2) + \mathrm{g}_1(t)]\}^{\frac{1}{2}}, \qquad (2.6)$$

for $t \in [0,T_1]$, where

$$g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}\int_0^t R(t)b(s)^2ds, \qquad (2.7)$$

where Γ is gamma function, $\Omega(v) = \int_{v_0}^v \frac{ds}{w(s)}, v_0 > 0, \Omega^{-1}$ is the inverse of Ω, and $t \in \mathbb{R}_+$ is such that $\Omega(2\mathrm{a}(\mathrm{t})^2) + \mathrm{g}_1(t) \in Dom(\Omega^{-1})$ for all $t \in [0,T_1]$.

(ii) Let $\beta = (0,\frac{1}{2}]$ and w satisfies the condition (2.3) with $q = z = 2$, where $z = \frac{1-\beta}{\beta}$, i. e. $\beta = \frac{1}{z+1}$. Let Ω, Ω^{-1} be as in part (i). Then

$$u(t) \leq e^t\{\Omega^{-1}[\Omega(2^{\mathrm{q}-1}\mathrm{a}(\mathrm{t})^2) + \mathrm{g}_2(t)]\}^{\frac{1}{q}}, \qquad (2.8)$$

for $t \in [0,T_1]$, where

$$g_2(t) = 2^{\mathrm{q}-1}K_z^q\int_0^t R(t)b(s)^2ds, \qquad (2.9)$$

$$K_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{\frac{1}{p}}, \alpha = \frac{z}{z+1}, p = \frac{z+2}{z+1} \qquad (2.10)$$

and $T_1 \in \mathbb{R}_+$ is such that $\Omega(2^{\mathrm{q}-1}\mathrm{a}(\mathrm{t})^2) + \mathrm{g}_2(t) \in Dom(\Omega^{-1})$ for all $t \in [0,T_1]$.

Lemma 2.2 [19] Let $0 \leq T \leq \infty$, $u(t), b(t), a(t), a'(t) \in C([0,T), R+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)u(s)ds, \qquad (2.11)$$

for $t \in [0,T)$ where $\beta > 0$ is constant. Then following hold:

(i) Suppose $\beta > \frac{1}{2}$, then

$$u(t) \leq (\sqrt{2})a(t)\exp\left[\frac{2\Gamma(2\beta-1)}{4^\beta}\int_0^t b(s)^2ds + t\right], \quad \text{for } t \in [0,T]. \qquad (2.12)$$

(ii) If $\beta = \frac{1}{z+1}$, for some $z \geq 1$, then

$$u(t) \leq (2^{q-1})^{\frac{1}{q}}\, a(t)\exp\left[\frac{2^{q-1}}{q}K_z^q\int_0^t b(s)^q ds + t\right], \qquad (2.13)$$

for $t \in [0,T]$, where K_z is defined as in (2.10), $q = z + 2$.

For detail proof of above two theorems see [16].

Lemma 2.3 ([3, 18], p. 152) (Bihari inequality) Let u and f be nonnegative defined on $\mathbb{R}_+$, let $w(u)$ be continuous nondecreasing function defined on $\mathbb{R}^+$, and $w(u) > 0$, on $(0,\infty)$. If

$$u(t) \leq k + \int_0^t f(s)w\big(u(s)\big)ds, \qquad (2.14)$$

for $t \in \mathbb{R}_+$ where k is nonnegative constant, for $0 \leq t \leq T$,

$$u(t) \leq G^{-1}\left[G(k) + \int_0^t f(s)ds\right], \qquad (2.15)$$

where

$$G(r) = \int_0^t \frac{ds}{w(s)}, r > 0, r_0 > 0$$

and G^{-1} is the inverse fucntion of G and $t_1 \in \mathbb{R}_+$ is chosen so that $G(k) + \int_0^t f(s)ds \in Dom(G^{-1})$ for all $t \in \mathbb{R}_+$ laying in the interval $0 \leq t \leq t_1$.

Lemma 2.4 Let (X,d) be a non-empty complete metric space with a contraction mapping $T: X \to X$. Then T admits a unique fixed point x^* in X (i. e. $T(x^*) = x^*$).

3. Existence and Uniqueness

In this section, we prove existence and uniqueness result for the problem (1.3)-(1.4). We first note that if $x \in C(J,\mathbb{R})$ is an absolutely continuous function satisfying (1.3)-(1.4), then

$$x(t) = \frac{x_0}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s, x(s), D^\alpha x(s)\big)ds. \qquad (3.1)$$

Theorem 3.1 *Assume that there exist* $M, L > 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq M|x-\bar{x}| + L|y-\bar{y}|, \qquad (3.2)$$

for each $x, \bar{x}, y, \bar{y} \in \mathbb{R}$. *If* $\left(\frac{Mb^\alpha}{\Gamma(\alpha)} + L\right) < 1$, *then the problem (1.3)-(1.4) has unique solution* $x \in C(J,\mathbb{R})$*on* J.

Proof. Consider a function $F: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ and defined by

$$F(z(t)) = f\left(t, \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J. \tag{3.3}$$

Let $z_1, z_2 \in C(J, \mathbb{R})$. Then we have

$$\begin{aligned} &\left|F(z_1(t)) - F(z_2(t))\right| \\ &\le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z_1(s) - z_2(s)| ds + L|z_1(t) - z_2(t)| \\ &\le \frac{Mt^\alpha}{\Gamma(\alpha)} \|z_1 - z_2)\|_\infty + L|z_1(t) - z_2(t)| \\ &\le \frac{Mt^\alpha}{\Gamma(\alpha)} \|z_1 - z_2)\|_\infty + L\|z_1 - z_2)\|_\infty. \end{aligned} \tag{3.4}$$

Hence

$$\left\|F(z_1(t)) - F(z_2(t))\right\|_\infty \le \left(\frac{Mt^\alpha}{\Gamma(\alpha)} + L\right) \|z_1 - z_2)\|_\infty, \tag{3.5}$$

for each $z_1, z_2 \in C(J, \mathbb{R})$. From the Banach fixed point theorem, Lemma 2.4, there exists a unique $z \in C(J, \mathbb{R})$ such that $z = F(z)$. Therefore

$$z(t) = f\left(t, \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J. \tag{3.6}$$

Set

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds. \tag{3.7}$$

This implies that $D^\alpha x(t) = z(t)$ and therefore

$$D^\alpha x(t) = f(t, x(t), D^\alpha x(t)), t \in J.$$

This shows that the function $x(t)$ satisfies the problem (1.3)-(1.4) and uniqueness of the solution follows from the unique existence of $z(t)$. This completes the proof of the theorem.

4. Uniqueness via Inequalites

In this section, we discuss the uniqueness of solution of the initial value problem (1.3)-(1.4).

Theorem 4.1 If $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies condition*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le M\Phi(|x - \bar{x}|) + L|y - \bar{y}|, \tag{4.1}$$

whre M *is positive constant,* $L \in (0,1)$ *and* Φ *is a continuous nondecreasing function on* $0 < u \le A$*, with* $\Phi(0) = 0$ *and*

$$\int_0^A \frac{du}{\Phi(u)}, \tag{4.2}$$

then the problem (1.3)-(1.4) has unique solution on J.

Proof. Let $x(t)$ and $y(t)$be two solutions of the problem (1.3)-(1.4). Then we have

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^\alpha x(s)) ds \tag{4.3}$$

and

$$y(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^\alpha y(s)) ds. \tag{4.4}$$

Hence we have

$$\begin{aligned} |x(t) - y(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s, x(s), D^\alpha x(s)) - f(s, y(s), D^\alpha y(s))\right| ds \\ &\le \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [M\Phi(|x(s) - y(s)|) + L|D^\alpha(x(s) - y(s))|] ds. \end{aligned} \tag{4.5}$$

But by hypothesis (4.1) for any $t \in [0,b]$ and any $x, y \in \mathbb{R}$,

$$\begin{aligned} |D^\alpha(x(t) - y(t))| &= | f(t, x(t), D^\alpha x(t)) - f(t, y(t), D^\alpha y(t))| \\ &\le M\Phi(|x(t) - y(t)|) + L|D^\alpha(x(t) - y(t))|. \end{aligned}$$

This implies

$$|D^\alpha(x(t) - y(t))| \le \frac{M}{1-L} \Phi(|x(t) - y(t)|). \tag{4.6}$$

Using above estimation in (4.5), we get

$$\begin{aligned} |x(t) - y(t)| &\le \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[M\Phi(|x(s) - y(s)|) + \frac{ML}{1-L} \Phi(|x(s) - y(s)|)\right] ds \\ &\le \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[M + \frac{ML}{1-L}\right] \Phi(|x(s) - y(s)|) ds \\ &\le \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M\left[1 + \frac{L}{1-L}\right] \Phi(|x(s) - y(s)|) ds \\ &\le \epsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{M}{1-L} \Phi(|x(s) - y(s)|) ds. \end{aligned} \tag{4.7}$$

Now an application of Lemma 2.3 to (4.7) which yields

$$|x(t)-y(t)| < \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\int_0^b (t-s)^{\alpha-1}ds\right]$$
$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\frac{(t-s)^{\alpha}}{-\alpha}|_0^b\right]$$
$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\left(\frac{b^{\alpha}}{\alpha}-\frac{(t-b)^{\alpha}}{\alpha}\right)\right]$$
$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha+1)(1-L)}(b^{\alpha}-(t-b)^{\alpha})\right], \tag{4.8}$$

where $\Phi(x)$ is primitive for $\frac{1}{\Phi(x)}$. We shall prove that the right-hand side of (4.8) tends toward zero as $\epsilon \to 0$. As $|x(t)-y(t)|$ is independent of ϵ, it follows that $x(t)=y(t)$, which we need. Let us remark that condition (4.2) implies $\Phi(\epsilon)\to -\infty$ for $\epsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\Phi(x)}$. Thus $\Phi^{-1}(x)\to 0$ as $x\to -\infty$. Consequently, $\epsilon \to 0$ in the inequality (4.8), the right-hand side tends toward zero. This completes the proof of the theorem.

Theorem 4.2 *If the function f is continuous and satisfies the conditionz*

$$|f(t,x,y)-f(t,\bar{x},\bar{y})| \le M|x-\bar{x}|+L|y-\bar{y}|, \tag{4.9}$$

for some positive constant M and $L\in(0,1)$ then the initial value problem (1.3)-(1.4) has unique solution in the interval J.

Proof. Let $x(t)$ and $y(t)$ be two solutions of the problem (1.3)-(1.4). Then we have

$$x(t)=\frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),D^{\alpha}x(s))ds \tag{4.10}$$

and

$$y(t)=\frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s),D^{\alpha}y(s))ds. \tag{4.11}$$

Therefore, using these (4.10), (4.11) and hypothesis (4.9), we have

$$|x(t)-y(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,x(s),D^{\alpha}x(s)\right)-f(s,y(s),D^{\alpha}y(s))\right|ds$$
$$\le \epsilon+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(|x(s)-y(s)|)+L|D^{\alpha}\left(x(s)-y(s)\right)|\right]ds. \tag{4.12}$$

But by again hypothesis (4.9) for any $t\in[0,b]$ and any $x,y\in\mathbb{R}$,

$$|D^{\alpha}(x(t)-y(t))| = |f(t,x(t),D^{\alpha}x(t)) - f(t,y(t),D^{\alpha}y(t))|$$
$$\le M(|x(t)-y(t)|)+L|D^{\alpha}(x(t)-y(t))|.$$

This implies

$$|D^{\alpha}(x(t)-y(t))| \le \frac{M}{1-L}(|x(t)-y(t)|). \tag{4.13}$$

Using (4.13) in (4.12), we obtain

$$|x(t)-y(t)| \le \epsilon+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(|x(s)-y(s)|)+\frac{ML}{1-L}(|x(s)-y(s)|)\right]ds$$
$$\le \epsilon+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M+\frac{ML}{1-L}\right](|x(s)-y(s)|)ds$$
$$\le \epsilon+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}M\left[1+\frac{L}{1-L}\right](|x(s)-y(s)|)ds$$
$$\le \epsilon+\int_0^t (t-s)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\frac{M}{1-L}(|x(s)-y(s)|)ds. \tag{3.14}$$

Now, (a) suppose that $\alpha > \frac{1}{2}$, then applying Leema 2.2 (i) to (4.14), we have

$$|x(t)-y(t)| < \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}\int_0^t (\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)})^2 ds+t\right]$$
$$< \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}(\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)})^2\int_0^t ds+t\right]$$
$$< \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}(\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)})^2 t+t\right], \tag{4.15}$$

for $t \in J$. Since ε was arbitrary, as $\varepsilon \rightarrow 0$ the inequality (4.13) implies that $x(t) = y(t)$ on J.

(b) Let $\alpha > \frac{1}{z+1}$ for some $z \geq 1$. Then by Lemma 2.2 (ii) to (4.14), again we have,

$$|x(t) - y(t)| < (2^{q-1})^{\frac{1}{q}} \epsilon \, exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t (\frac{1}{\Gamma(\alpha)} \frac{M}{(1-L)})^2 ds + t\right]$$

$$< (2^{q-1})^{\frac{1}{q}} \epsilon \, exp\left[\frac{2^{q-1}}{q} K_z^q (\frac{1}{\Gamma(\alpha)} \frac{M}{(1-L)})^2 t + t\right], \tag{4.16}$$

for $t \in [0, b)$ where K_z is defined by (2.10). Since ε was arbitrary in (4.16), implies that $x(t) = y(t)$ as $\epsilon \rightarrow 0$. This completes the proof of the theorem.

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