

# On Extracting Properties of Lie Groups from Their Lie Algebras

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**Abstract** The study of Lie groups and Lie algebras is very useful, for its wider applications in various scientific fields. In this paper, we introduce a thorough study of properties of Lie groups via their lie algebras, this is because by using linearization of a Lie group or other methods, we can obtain its Lie algebra, and using the exponential mapping again, we can convey properties and operations from the Lie algebra to its Lie group. These relations help in extracting most of the properties of Lie groups by using their Lie algebras. We explain this extraction of properties, which helps in introducing more properties of Lie groups and leads to some important results and useful applications. Also we gave some properties due to Cartan's first and second criteria.

**Keywords** Lie group, Lie algebra, Exponential mapping, Linearization Killing form, Cartan's criteria

## 1. Introduction

The Properties and applications of Lie groups and their Lie algebras are too great to overview in a small paper. Lie groups were greatly studied by Marius Sophus Lie (1842 – 1899), who intended to solve or at least to simplify ordinary differential equations, and since then the research continues to disclose their properties and applications in various scientific fields [1, 2, 5, 6]. Lie groups have algebraic properties derived from their group axioms, and geometric properties due to identification of their group operations with points in a topological space, namely every Lie group is a differentiable manifold [3, 4, 7]. Almost all of the Lie groups encountered in many applications are matrix groups, and this simplifies their study. This simplification can be done by linearizing the Lie group in the neighborhood of its identity, which results in what is called a Lie algebra [1]. The Lie algebra retains most of, but not all of the properties of the original Lie group. Also most of the Lie group properties can be recovered by the inverse of the linearization operation, namely, the exponential mapping. A Lie algebra is a linear vector space, and this makes it easy to study it by using tools of linear algebra rather than studying the original Lie group with its complicated structure. In this manner we can extract properties of a Lie group from its Lie algebra, and this opens the door to many scientific applications, for instance in mathematics and physics.

## 2. Lie Group

### 2.1. Definition (Lie group)

A Lie group  $G$  is an abstract group and a smooth  $n$ -dimensional manifold so that multiplication  $G \times G \rightarrow G : (a, b) \rightarrow ab$  and inverse  $G \rightarrow G : a \rightarrow a^{-1}$  are smooth.

### 2.2. Examples

- The general linear group  $GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \det A \neq 0\}$  is an open set of  $\mathbb{R}^{n \times n}$ , it is a manifold with the two operations, matrix multiplication and matrix inverse which are  $C^\infty$  mappings. So it is a Lie group.
- The complex linear group  $GL(n, \mathbb{C})$ , the group of nonsingular  $n \times n$  matrices is also a Lie group.
- We can take the two subsets of the general linear group: the special linear group  $SL(n, \mathbb{R})$  which is the group of matrices of determinant 1 and the other is the orthogonal group  $O(n)$  consisting of all matrices  $A$  satisfying  $A^T A = I$ , these two subsets are also Lie groups.

Matrix groups are very important examples of Lie groups because of their various scientific applications. There are many examples of Lie groups different from matrix groups.

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### 3. Lie Algebra

#### 3.1. Definition (Lie Algebra)

A Lie algebra over a field  $K$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) is a vector space  $V$  over  $K$  with a skew – symmetric  $k$ - bilinear form (the Lie bracket)  $[\cdot, \cdot]: V \times V \rightarrow V$ , which satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (3.1)$$

For all  $X, Y, Z \in V$

The tangent space  $T_e G$  at the identity of a Lie group canonically has the structure of Lie algebra. This Lie algebra encodes in it much information about the Lie group. So we have to define the Lie algebra structure on  $T_e G$  and this can be shown if we identified this structure with the Lie algebra for instance for some classical groups.

#### 3.2. Example

Since  $GL(n, \mathbb{R})$  is an open subset of  $\mathbb{R}^{n \times n}$  (the vector space of all  $n \times n$  real matrices), the tangent space to  $GL(n, \mathbb{R})$  at the identity  $I$  is  $\mathbb{R}^{n \times n}$  itself.

Also the tangent space  $T_I(SL(n, \mathbb{R}))$  is the subspace of  $\mathbb{R}^{n \times n}$  consisting of all  $n \times n$  matrices of trace 0.

And for the group of orthogonal matrices  $O(n)$ , the tangent space at the identity is consisting of skew – symmetric matrices. These are some examples of Lie algebras of some groups of matrices considered as tangent spaces at the identity.

### 4. Linearization of a Lie Group

The Lie group may have a complicated structure which make it difficult to study it through some mathematical tools, instead of this we linearize the group in the neighborhood of its identity, that is by getting a vector space at the identity, this vector space is its Lie algebra. Therefore, Linearization of Lie group makes it possible to study it by using tools of Linear algebra. Moreover most of the Lie group properties can be recovered by the inverse of the linearization operation, that is by the exponential mapping. The following diagram expresses how to go from the Lie group to its Lie algebra and vice versa by using these two operations:

$$\text{Lie group} \xrightleftharpoons[\text{Exponentiation}]{\text{Linearization}} \text{Lie algebra}$$

#### 4.1. How To Linearize a Lie Group?

In order to linearize a Lie group to get its Lie algebra we need an explicit parametrization of the underlying manifold and also we need an available expression for the group composition law. For matrix groups the group composition law is matrix multiplication and every matrix group is a closed subset of the general linear group, so we can do

linearization immediately as we see in the following example:

#### 4.2. Example

To linearize the Lie group

$$SL(2, \mathbb{R}) = \{M \in GL(2, \mathbb{R}), \det M = 1\}$$

We parametrize the group by:

$$(a, b, c) \rightarrow M(a, b, c) = \begin{bmatrix} 1+a & b \\ c & (1+bc)/(1+a) \end{bmatrix} \quad (4.1)$$

then we let  $(a, b, c)$  to become infinitesimals:

$$(a, b, c) \rightarrow (\delta a, \delta b, \delta c) \rightarrow M(\delta a, \delta b, \delta c) = \begin{bmatrix} 1+\delta a & \delta b \\ \delta c & (1+\delta b \delta c)/(1+\delta a) \end{bmatrix} \quad (4.2)$$

so the basis vectors in the Lie algebra  $X_a, X_b$  and  $X_c$  are the coefficients of the first order infinitesimals:

$$(\delta a, \delta b, \delta c) \rightarrow I_2 + \delta a X_a + \delta b X_b + \delta c X_c = \begin{bmatrix} 1+\delta a & \delta b \\ \delta c & 1-\delta a \end{bmatrix} \quad (4.3)$$

We get the basis vectors from (4-1) as follows:

$$\left. \begin{aligned} X_a &= \frac{\partial M}{\partial a} \bigg|_{(0,0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ X_b &= \frac{\partial M}{\partial b} \bigg|_{(0,0,0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ X_c &= \frac{\partial M}{\partial c} \bigg|_{(0,0,0)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned} \right\} \quad (4.4)$$

These are the basis vectors for the Lie algebra  $sl(2, \mathbb{R})$  in our example.

### 5. Exponential Mapping

Suppose  $V$  is a finite dimensional vector space equipped with a complete norm  $\|\cdot\|$  over the field  $\mathbb{R}$  or  $\mathbb{C}$ . If  $End(V)$  denotes the algebra of linear automorphisms on  $V$ , and  $GL(V)$  the general linear group. Then for  $V = \mathbb{R}^n$ ,  $End(V)$  is identified with  $M_n(\mathbb{R})$  ( $n \times n$  matrices) and  $GL(V) = GL(n, \mathbb{R})$ .

#### 5.1. Definition (Matrix Group)

A matrix group is a closed subgroup of  $GL(V)$  (the general linear group), where  $V$  is a finite dimensional vector space.

### 5.2. Definition (Exponential map)

The exponential map on  $End(V)$  is defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \quad A \in End(V) \quad (5.1)$$

If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , then the exponential map is defined as:

$$\exp: \mathfrak{g} \rightarrow G \text{ where } \exp(X) = \phi_X(1) \text{ with } \phi_X'(0) = X, \\ \forall X \in \mathfrak{g}$$

### 5.3. Definition (One – Parameter Subgroup)

A one – parameter subgroup of a Lie group  $G$  is a continuous homomorphism:  $\alpha: (\mathbb{R}, +) \rightarrow G$

### 5.4. Example

For the Lie groups  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$ , we have  $\exp(A) = e^A$ ,  $e^{(t,s)} = e^{tA} e^{sA}$ , i.e.  $\phi(t) = e^{tA}$  is a one parameter group. Furthermore  $\phi'(0) = A$  and hence  $\exp(A) = \phi(1) = e^A$ .

### 5.5. Proposition [2]

For  $V$  a finite dimensional normed vector space and  $A \in End(V)$ , the map  $t \rightarrow \exp(tA)$  is a one – parameter subgroup of  $GL(V)$ . In particular  $(\exp(A))^{-1} = \exp(-A)$ .

### 5.6. Theorem [2]

Every one – parameter subgroup  $\alpha: \mathbb{R} \rightarrow End(V)$  is of the form  $\alpha(t) = \exp(tA)$  for some  $A \in End(V)$ .

### 5.7. Definition (The Infinitesimal Generator)

The element  $A \in End(V)$  is called the infinitesimal generator of the one – parameter subgroup  $t \rightarrow \exp(tA)$ .

### 5.8. Lemma

The exponential map is absolutely convergent, hence convergent on all of  $End(V)$ . Hence it defines an analytic automorphism on  $End(V)$

**Proof**

$$\left\| \sum_{n=0}^{\infty} \frac{A^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = \exp(\|A\|)$$

### 5.9. Lemma [2]

The identity map on  $End(V)$  is the derivative at 0 of  $\exp: End(V) \rightarrow End(V)$  that is,  $d\exp(0) = Id$ .

### 5.10. Theorem (The Lie Algebra of a Matrix Group)

For a matrix group  $G \subseteq GL(V)$ , the set  $\mathfrak{g} = \{A \in End(V) : \exp(tA) \in G \text{ for all } t \in \mathbb{R}\}$  is a Lie algebra, called the Lie algebra of  $G$ .

Sketch of the Proof:

The theorem can be proved by showing that:

$\exp(t[A, B]) = \exp([tA, B]) \in G$  for all  $t \in \mathbb{R}$ . Thus the commutator  $[A, B] \in \mathfrak{g}$ .

### 5.11. Computing Lie Algebras

For matrix groups we can use some methods to find their Lie algebras. For instance we can use the following proposition for this purpose.

### 5.12. Proposition

Let  $\beta(.,.)$  be a continuous bilinear form on the vector space  $V$  and set  $G = \{h \in GL(V) : \forall x, y \in V, \beta(hx, hy) = \beta(x, y)\}$ , then the Lie algebra is  $\mathfrak{g} = \{A \in End(V) : \forall x, y \in V, \beta(Ax, y) + \beta(x, Ay) = 0\}$

**Proof**

Let  $A \in \mathfrak{g}$ , then  $\beta(\exp(tA)x, \exp(tA)y) = \beta(x, y)$  for all  $x, y \in V$ .

Differentiating with respect to  $t$ :

$$\beta(A \exp(tA)x, \exp(tA)y) + \beta(\exp(tA)x, A \exp(tA)y) = 0,$$

then at  $t=0 \Rightarrow \beta(Ax, y) + \beta(x, Ay) = 0$ .

Conversely: for all  $x, y \in V$ ,  $\beta(Ax, y) + \beta(x, Ay) = 0$ .

Then for  $f(t) = \beta(\exp(tA)x, \exp(tA)y)$ ,  $f'(t) = 0$ , therefore  $f$  is constant function of a value  $\beta(x, y)$  at  $t=0$ . It follows that:

$$\exp(tA) \in G \text{ for all } t, \text{ that is } A \in \mathfrak{g}$$

## 6. Exponentiation

We can reverse the linearization operation by using the exponential map so as to recover most of the properties of the Lie group from its Lie algebra. We can see the behavior of elements near the identity of the Lie group. Suppose  $X$  is an operator in a Lie algebra, and let  $\delta$  be a small real number. Then  $I + \delta X$  represents an element in the Lie group close to the identity. Far from the identity by iterating this group operation many times we get the exponential function:

$$\lim_{k \rightarrow \infty} \left( I + \frac{1}{k} X \right)^k = \sum_{n=0}^{\infty} \frac{X^n}{n!} = \exp(X) \quad (6.1)$$

### 6.1. Example

On taking the Lie group  $G = SL(2, \mathbb{R})$ , we can show how to exponentiate an operator in its Lie algebra to get the result in the group  $G$  as follows:

given the basis vectors  $X_a, X_b, X_c \in sl(2, \mathbb{R})$  (the Lie algebra), let us take the operator

$$X = aX_a + bX_b + cX_c = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \text{ then}$$

$$\begin{aligned} \exp(X) &= \exp(aX_a + bX_b + cX_c) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}^n \\ &= I_2 \cosh \theta + X \frac{\sinh \theta}{\theta} \\ &= \begin{bmatrix} \cosh \theta + \frac{a \sinh(\theta)}{\theta} & \frac{b \sinh(\theta)}{\theta} \\ \frac{c \sinh(\theta)}{\theta} & \cosh \theta - \frac{a \sinh(\theta)}{\theta} \end{bmatrix} \quad (6.2) \end{aligned}$$

where  $\theta^2 = a^2 + bc$ .

In fact  $X^0 = I_2, X^1 = X, X^2 = \theta^2 I_2$  so  $X^3 = X^2 X^1 = \theta^2 X$  is proportional to  $X$  and  $X^4 = X^3 X^1$  is proportional to the identity and so on.

We can put  $F(X) = f_0(a, b, c)X^0 + f_1(a, b, c)X^1$  where  $f_0$  &  $f_1$  are functions of the invariants of the matrix, that is  $\theta^2 = a^2 + bc$  (Cayley – Hamilton theorem).

So that

$$\exp(X) = f_0(\theta^2)I_2 + f_1(\theta^2)X \quad (6.3)$$

where

$$f_0(\theta^2) = I + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \dots = \cosh \theta \quad (6.4)$$

$$f_1(\theta) = I + \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \frac{\theta^6}{7!} + \dots = \frac{\sinh(\theta)}{\theta} \quad (6.5)$$

## 7. Isomorphism on Lie Groups and Lie Algebras

Isomorphic Lie groups have isomorphic Lie algebras, but, two Lie groups with isomorphic Lie algebras need not be isomorphic as we see in the following example.

### 7.1. Example

The groups  $SO(2, 1)$  &  $SU(1, 1)$  have respectively the Lie algebras

$$so(2, 1) = \begin{bmatrix} 0 & a_3 & a_2 \\ -a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \quad \& \quad su(1, 1) = \frac{i}{2} \begin{bmatrix} b_3 & ib_1 + b_2 \\ ib_1 - b_2 & -b_3 \end{bmatrix}.$$

The Lie algebras are isomorphic because there is a 1:1 correspondence between them, but the groups have a 2:1 correspondence, that is why they are not isomorphic.

## 8. Some Properties

### 8.1. Definition (Ideals and Simple Lie Algebras)

An ideal  $I$  of a Lie algebra  $\mathfrak{g}$  is a vector space of  $\mathfrak{g}$  such that  $[a, b] \in I, \forall a \in I \text{ and } b \in \mathfrak{g}$

A simple Lie algebra is the one which has no proper ideal. Also a semisimple Lie algebra is the one which is a direct sum of simple Lie algebras.

### 8.2. Example

Let  $sl_n(\mathbb{C})$  be the set of all  $n \times n$  matrices of trace 0.  $sl_n(\mathbb{C})$  is an ideal of  $gl_n(\mathbb{C})$  which is nonzero. So  $gl_n(\mathbb{C})$  is not simple.

### 8.3. Definition (The Adjoint Representation)

Let  $\mathfrak{g}$  be a Lie algebra over  $k$ . A representation of  $\mathfrak{g}$  is a Lie algebra homomorphism  $\rho: \mathfrak{g} \rightarrow gl(n, k)$  for some  $n$  called the degree of the representation. We define a mapping  $ad X$  from a Lie algebra to itself by  $ad X: Y \rightarrow [X, Y]$ . The mapping  $X \rightarrow ad X$  is a representation of the Lie algebra called the adjoint representation. It is an automorphism.

### 8.4. Cartan Subalgebra $\mathfrak{h}$

A Cartan subalgebra  $\mathfrak{h}$  for a Lie algebra  $\mathfrak{g}$  is an abelian, diagonalizable subalgebra which is maximal under set inclusion. Its dimension is the rank of  $\mathfrak{g}$ .

All Cartan Lie algebras of a Lie algebra  $\mathfrak{g}$  are conjugate under automorphisms of  $\mathfrak{g}$ , and they have the same dimension.

Define the basis  $\{H_1, \dots, H_r\}$  for  $\mathfrak{h}$ . Since  $\mathfrak{h}$  is abelian,  $[H_i, H_j] = 0$  for all  $i, j$ .

If we extend this basis for  $\mathfrak{h}$  to a basis for  $\mathfrak{g}$ , and then we get a much simpler basis for  $\mathfrak{g}$  with convenient commutator relations. This can be used in many applications of Lie algebras, especially in classification machinery.

### 8.5. The Killing Form

For the Lie algebra  $\mathfrak{g}$ , if  $X, Y \in \mathfrak{g}$ , we define the Killing form by  $k(X, Y) = \text{tr } \text{adj } X \text{ adj } Y$ , it resembles a scalar product for elements of a Lie algebra itself. It is invariant under all automorphisms of  $\mathfrak{g}$ .

The Killing form is an effective tool in calculations in Lie algebra and symmetric spaces and in describing some properties due to some Cartan's theorems.

### 8.6. Cartan's First Criterion [8]

A Lie algebra  $\mathfrak{g}$  is solvable if and only if its Killing form vanishes identically on the derived Lie algebra  $\mathfrak{g}'$ .

It is easy to see that both solvability and vanishing of the Killing form remain unchanged under complexification for a real Lie algebra  $\mathfrak{g}$ ; thus we may take  $\mathfrak{g}$  complex.

### 8.7. Cartan's Second Criterion [8]

A Lie algebra  $\mathfrak{g}$  is semisimple if and only if its dimension is positive and its killing form is non-degenerate.

(Non-degenerate means: If for some  $X_0$  in  $\mathfrak{g}$  the value of the killing form is 0 for all  $Y$  in  $\mathfrak{g}$ , then  $X_0$  is 0).

Just as for the first criterion we may assume that  $\mathfrak{g}$  is complex, since both semisimplicity and non-degeneracy of the killing form are unchanged by complexification.

## 9. Conclusions

1. The exponential operation provides a natural parametrization of the Lie group in terms of linear quantities. This function maps the linear vector space – the Lie algebra – to the geometric manifold that parameterizes the Lie group.
2. It is not always possible to map a Lie algebra onto the entire Lie group through a single mapping of the form  $\exp(X)$ .
3. Since the exponential map takes elements of the Lie algebra to the geometric manifold that parametrizes the Lie group, we expect not to be able to extract all properties of the Lie group from its Lie algebra, but

most of these properties can be retained.

4. Since isomorphic Lie groups have isomorphic Lie algebras, research can go on studying joined properties of Lie groups using their isomorphic Lie algebras.
5. The killing form is an effective tool in describing some important properties of the Lie algebra and in carrying some calculations on it.

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## REFERENCES

- [1] Robert Gilmore, Lie Groups, Physics, and Geometry, Cambridge University Press, 2008.
- [2] Loring W. Tu,, An Introduction to Manifolds, Springer, 2008.
- [3] William M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press Inc. New York, 1975.
- [4] Jimmie Lawson, Matrix Lie Groups and Control Theory, Summer 2007.
- [5] Sigurdur Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press INC. 1978.
- [6] William Fulton and Joe Harris. Representation theory: a first course, volume 129. Springer, 1991.
- [7] Karin Erdmann and Mark J Wildson. Introduction to Lie algebras. Springer, 2006.
- [8] Hans Samelson, Notes on Lie Algebras, third edition.