

# Refined Iterative Method for Solving System of Linear Equations

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**Abstract** In this paper, the refined iterative method namely, refinement of generalized Gauss-Seidel (RGGS) method for solving systems of linear equations is studied. Sufficient conditions for convergence are given and some numerical experiments are considered to show the efficiency of the method. The result shows that RGGS method converges if the coefficient matrix is diagonally dominant (DD) or an M- matrix for any initial vectors, moreover it is more efficient than the other methods Refinement of generalized Jacobi (RGJ) and successive-over relaxation (SOR) methods, considering their performance, using parameters such as time to converge, number of iterations required to converge and level of accuracy.

**Keywords** Generalized Gauss-Seidel Method, M-matrix, Row strictly diagonally dominant matrix, Convergence

## 1. Introduction

Consider the linear systems of equations

$$AU = b \tag{1.1}$$

where  $A = (a_{ij})$  be an  $N \times N$  non- singular matrix with non-zero diagonal elements,  $U$  and  $b$  are  $N$ -dimensional vectors. For the determination of the  $N$ -dimensional solution vector  $U$  of Eq. (1.1) using a stationary first order iterative method, which can be written in the form of

$$U^{(n+1)} = BU^{(n)} + C \tag{1.2}$$

The Gauss-Seidel method is more efficient if it combined with other method, [1]. It has also been proved that if  $A$  is strictly diagonally dominant or irreducible diagonally dominant or symmetric positive definite matrix, the Gauss-Seidel method converges for any initial approximation  $U^{(0)}$ . On the other hand [1] developed the method called generalized Gauss-Seidel method and the result shows that the method is more efficient than conventional Gauss-Seidel method. Author [2], indicated that the basic idea behind the first order stationary iterative method is how to write Eq. (1.2) and the choice of initial approximation to guarantee the convergence of the method. In this paper, the generalized Gauss-Seidel method is refined and compared its efficiency with the other methods.

Some of the basic definition of terms used in the this paper given here as under

**Definition 1:** A matrix  $A$  is said to be an M-matrix if it satisfies the following four properties

- i.  $a_{ii} > 0$ , for  $i = 1(1)N$
- ii.  $a_{ij} \leq 0$ , for  $i \neq j$ ,  $i, j = 1(1)N$
- iii.  $A$  is non-singular
- iv.  $A^{-1} \geq 0$

**Definition 2:** A banded matrix is a square matrix with zeros after “ $m$ ” elements above and below the main diagonal, where  $m$  is less than the size of the matrix (i.e if the matrix is  $N \times N$  then  $m < N$ ). In this case where bandedness mater, “ $m$ ” is usually significantly less than  $N$ .

## 2. Preliminary

Considering the system of equations given in Eq. (1.1) and using splitting procedures in [1], we obtain:

$$A = T_m - E_m - F_m \tag{2.1}$$

where  $T_m = (t_{ij})$  be a banded matrix of band width  $2m+1$  defined as

$$t_{ij} = \begin{cases} a_{ij} & , 0 \leq |i - j| \leq m \\ 0 & , otherwise \end{cases}$$

$E_m$  and  $F_m$  are strictly lower and upper triangular part of the matrix  $A - T_m$  respectively, and defined as follows.

$$T_m = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m+1} & & & \\ \vdots & \ddots & & \ddots & & \\ a_{m+1,1} & & & \ddots & & a_{N-m,N} \\ & \ddots & & & \ddots & \vdots \\ & & a_{N,N-m} & \cdots & & a_{N,N} \end{bmatrix},$$

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ii) If A is an M- matrix, then  $\|\bar{B}_{GGs}^m\|_\infty \leq \|B_{GGs}^m\|_\infty < 1$ .

**Proof:-i)** By the convergence Theorem of [4], we have

$$\rho(B_{GGs}^m) \leq \|B_{GGs}^m\|_\infty < 1$$

where  $\rho(B_{GGs}^m)$  is the spectral radius of generalized Gauss-Seidel method. Again by Theorem 5, we have

$$\|(T_m - E_m)^{-1}F_m\|_\infty < 1. \tag{4.1}$$

As a result of Eq. (4.1), we have

$$\begin{aligned} \|[(T_m - E_m)^{-1}F_m]^2\|_\infty &= \|(T_m - E_m)^{-1}F_m\|_\infty^2 \\ &\leq \|(T_m - E_m)^{-1}F_m\|_\infty < 1. \end{aligned}$$

Therefore,  $\|\bar{B}_{GGs}^m\|_\infty \leq \|B_{GGs}^m\|_\infty < 1$ .

ii) Similar to (i) above, one can complete the proof the Theorem.

**Remark:** - We observe that the iterative matrix of refinement of generalized Gauss-Seidel is the square of generalized Gauss-Seidel iterative matrix. i.e.  $\bar{B}_{GGs}^m = [B_{GGs}^m]^2$ . As it can be easily realized  $\rho(\bar{B}_{GGs}^m) = [\rho(B_{GGs}^m)]^2$ . If GGS method converges  $\rho(B_{GGs}^m) < 1$ , then  $\rho(\bar{B}_{GGs}^m) \leq \rho(B_{GGs}^m)$ . Thus, if the GGS and RGGS method converge, the RGGS method converges faster than the GGS method.

### 5. Numerical Experiments

The major factors to be considered in comparing different numerical methods are the accuracy of numerical solution and its computational time, [9]. The same author, indicated that the comparison of numerical methods is not simply because of their performance may depend on the characteristic of the problem at hand. It should also be noted that there are other factors to be considered, such as stability, versatility, proof against run-time error, and so on which are being considered in most of the MATLAB built-in routines, [10]. Upon this, the efficiency of RGGS was compared with SOR and RGJ of [5] by considering two model problems in which it reduced to system of linear equations of which its coefficient matrix is either M-matrix or diagonal dominant matrix and also it illustrate the theory developed in this paper. Data about iteration number and computational times (in seconds) obtained using RGGS, SOR and RGJ is used for the analysis of the result.

1. Consider the system of linear equations, which obtained from the finite difference approximation of the mixed boundary value problem considered by Jain and et al [8] on page 106.

$$\begin{bmatrix} 2 & -3 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -3 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 2/3 \\ 3 \\ -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

2. Consider a system of linear equations, which is reduced from the boundary value problem,  $\nabla^2 u = 1$ , on  $|x| \leq 1$ ,  $|y| \leq 1$  where  $u = 0$  on the boundary. By applying Galerkin Method with triangular element and considering the symmetry of the boundary, Jain and et al [8].

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -1/4 \\ -1/4 \\ -1/4 \end{bmatrix}$$

The coefficient matrix of the first example is an M-Matrix whereas the second one is diagonal dominant for which the mentioned methods are convergent for any initial vector. Results produced by the two linear systems of equation are given in the Tables 1 and 2 respectively.

**Table 1.** Linear System of Equations of Order 6X6

Numerical methods	Number of iteration	Computational time (in seconds)
SOR	14	0.014783
RGJ	10	0.010498
RGGS	6	0.003789

**Table 2.** Linear Systems of Equations of Order 4 X4

Numerical methods	Number of iteration	Computational time (in seconds)
SOR	9	0.034219
RGJ	11	0.014208
RGGS	6	0.002961

### 6. Discussion and Conclusions

The Refinement of Generalized Gauss-Seidel method for solving linear systems of equation has been presented. Two examples, which obtained from boundary value problems and reduced to systems of linear equations of 6 x6 and 4 x 4 by finite difference approximation and finite element method respectively, were studied using MATLAB version 7.60(R2008a) software. The results obtained by Refinement of generalized Gauss-Seidel are compared with that of Refinement of generalized Jacobi and Successive-Over Relaxation as presented by Table 1 and Table 2. The analysis of results shows that RGGS method takes shorter time, 0.003789 seconds and 0.002961 seconds for the 6 x 6 and 4 x 4 linear equations respectively. In terms of number of iterations required to converge to the exact solution, the RGGS method takes about 6 iteration for each problem as compared to other methods, namely; SOR (14 iteration for 6x6 and 9 iteration for 4x4 system of equations) and RGJ (10 iteration for 6x6 and 11 iteration for 4x4 system of equations) with  $5 \times 10^{-6}$  error of tolerance. On the other hand, methods

which register small number of iteration demands less computer storage to store its data, [3], as a result RGGGS demands less computer storage to store its data relative to SOR and RGJ. Thus, in all the parameters considered in this paper, i.e. CPU runtime, number of iteration and memory storage, the Refinement of generalized Gauss- Seidel is more efficient than the other methods considered when the coefficient matrix is either an M-matrix or diagonal dominant matrix.

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