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Comparative Notes on Banach Principle for Semifinite von Neumann Algebras (W* - Algebras) and JW-Algebras

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Abstract The objective of the paper is to give a comparative survey notes on non-commutative extension of the Banach Principle for L^∞ that was suggested in [3], [7], which extend the results in [8] to the case of JW-algebras without direct summand of the type I_2. We discuss relationships among the conditions $(BSCNV(A_1)),(BSCNT(A_1)),$ and $(BSCLS(A_1))$ in JW-algebras as discussed in the case of the *-algebras. We introduced the notion of uniform equicontinuity for sequences of functions with values in the space of measurable operators and present a non – commutative version of the Banach Principle for L^∞. We established the Banach Principle for semi-finite $JW-a\lg ebras$ without direct summand of type I$_2$, which was the extension of the results of Chilin and Litvinov on the Banach Principle for semi-finite von Neumann algebras. The results in this paper has shown how Banach Principle for semi-finite Von Neumann (W*-algebras) algebras was extended to the case of $JW-a\lg ebras$ without direct summand of type I$_2$.

Keywords Von Neumann algebras, Measure topology, Jordan operator algebras, Almost uniform convergence, Banach Principle, *-algebra of $\tau-$measurable operators affiliated to a semi-finite von Neumann algebra

1. Introduction

Let $(\Omega,\sum,\mu)$ be a probability space. Denote by $\mathcal{L}$ = $\mathcal{L}(\Omega,\mu)$, the set of all (classes of) complex-valued measurable functions on Ω. Let τ_μ stand for the measure topology in $\mathcal{L}$. The classical Banach principle can be stated as follows:

Classical Banach Principle. Let $(X,\|.\|)$ be a Banach space, and let $a_n:(X,\|.\|)\to(\mathcal{L},\tau_\mu)$ be a sequence of continuous linear maps. Consider the following properties:

(i) the sequence $\{a_n(x)\}$ converges almost everywhere (a.e) for every $x\in X$;

(ii) $\hat{a}(x)(\omega)=\sup_n|a_n(x)(\omega)|<\infty$ a.e for every $x\in X$;

(iii) holds, and the maximal operator $\hat{a}:(X,\|.\|)\to(\mathcal{L},\tau_\mu)$ is continuous at 0;

(iv) the set $\{x\in X:\{a_n(x)\}\,converges\ a.e.\}$ is closed in X.

Then the implications $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)$ are always true. If in addition, there exists a dense subset $D\subset X$, such that the sequence $\{a_n(x)\}$ converges a.e. for every $x\in D$, then all the four conditions (i) – (iv) above are equivalent.

The Banach Principle above was often applied in the case $X=(L^p,\|.\|_p)$, where $1\le p<\infty$. However, in the case p = ∞ the uniform topology on L^∞ appears to be too strong for the classical Banach Principle to be effective in L^∞. For example, continuous functions are not uniformly dense L^∞.

Bellow and jones [3], using the fact that the unit ball $L_1^\infty=\{x\in L^\infty:\|x\|_\infty\le 1\}$ is complete in τ_μ, suggested to consider the measure topology on L^∞ by replacing $(X,\|.\|)$ by (L_1^∞,τ_μ), since L_1^∞ is not a linear space, however, some geometrical implications do occurred but was resolved by Bellow and Jones. A non-commutative version of the Banach Principle for L^∞ was proposed by Chilin and Litvinov, while the non-commutative notions of Banach Principle for measurable operators affiliated to a semi-finite von Neumann algebra (W*-algebra) were established by Goldstein, M; Litvinov, S and Litvinov, S; Mukhamedov, F. Then it was refined and applied in [5], [6],

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[13]. In [5] the notion of uniform equicontinuity of a sequence of functions into $L(M,\tau)$ was introduced. The objective of the paper is to give a comparative survey notes on non-commutative extension of the Banach Principle for L^{∞} that was suggested in [3], [7], which extend the results in [8] to the case of JW-algebras without direct summand of the type I_2.

2. Preliminaries

Let M be a semi-finite von Neumann algebra of bounded operators acting on a complex Hilbert space H, and let $B(H)$ denote the algebra of all bounded linear operators on H. A densely-defined closed operator x in H is said to be *affiliated* to M if $y'z = zy', z \in M$. We denote by $P(M)$ the complete lattice of projections in M. Let τ be a faithful normal semi-finite trace on M Denote by $e^{\perp} = I - e$ the orthogonal complemented projection for the projection $e \in P(M)$, where I is the identity of M. An operator x affiliated to M is called $\tau - measurable$ if for every $\varepsilon > 0,$ there exist a projection $e \in P(M)$ with $\tau(e^{\perp}) \le \varepsilon$ such that eH belongs to the domain of the operator x. Let $L(M,\tau)$ the set of all $\tau - measurable$ operator affiliated to M. Denote $\|.\|$ the uniform norm in $B(H)$.

If we set

$$V(\varepsilon,\delta) = \{x \in L(M,\tau) : \|xe\| < \delta \ \ for \ some \ e \in P(M) \ with \ \tau(e^{\perp}) \le \varepsilon\}$$

then, the topology t_{τ} defined on $L(M,\tau)$ by the family $\{V(\varepsilon,\delta) : \varepsilon > 0, \delta > 0\}$ of neighborhoods of zero is called the *measure topology* ([14], [15])

Theorem 1: $(L(M,\tau), t_{\tau})$ is a complete metrizable topological *-algebra

Proof: see ([14], [15]) for the details of the proof.

Let A be a semifinite $JW - subal\,gebra$ of $B(H)_{SA}$ without a direct summand of type I_2 and $P(A)$ be the complete lattice of projections in A, and τ be a faithful normal semifinite trace on $A.$ Let $M = M(A)$ be the von Neumann enveloping algebra of the Jordan algebra $A.$ Then τ can be uniquely extended to a faithful normal semifinite trace on $M,$ for which we will use the same symbol $\tau.$ A self adjoint operator $x \in L(M,\tau)$ is called affiliated to a $JW - a\lg ebra\, A,$ if all its spectral projections belong to $A.$ An operator x affiliated to A is called $\tau - measurable$ if for all $\varepsilon > 0$ there exist $e \in P(A)$ with $\tau(e^{\perp}) \le \varepsilon$ such that eH belongs to the domain of the operator x. Let $L(A,\tau)$ be the set of all $\tau - measurable$ operators affiliated to A.

Remark: A sequence $\{y_n\} \subset L(M,\tau)$ is said to converge almost uniformly (a.u) to $y \in L(M,\tau)$ if for all $\varepsilon > 0,$ there exist $e \in P(M)$ with $\tau(e^{\perp}) \le \varepsilon$ such that $\|(y - y_n)e\| \to 0.$

Proposition 1: For $\{y_n\} \subset L(M,\tau)$ the following conditions are equivalent

(i) $\{y_n\}$ converges a.u. in $L(M,\tau)$;

(ii) For all $\varepsilon > 0,$ there exist $e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ such that $\|(y_m - y_n)e\| \to 0$ as $m,n \to \infty$

Proof: see [7] for details

The following theorem is a non-commutative notion of Riesz theorem.

Theorem 2: If $\{y_n\} \subset L(M,\tau)$ and $y = t_{\tau} - \lim_{n\to\infty} y_n$, then $y = a.u. - \lim_{k\to\infty} y_{n_k}$ for some subsequence $\{y_{n_k}\} \subset \{y_n\}$

The proof of this theorem can be seen in [9] and [15].

A sequence $\{y_n\} \subset L(A,\tau)$ is said to converge bilaterally with square almost uniformly (b.s.a.u.) to $y \in L(A,\tau)$ if given $\varepsilon > 0$ there is $e \in P(A)$ with $\tau(e^{\perp}) < \varepsilon$ s.t. $\|e(y - y_n)^2 e\| \to 0.$

Then the following proposition holds;

Proposition 2: For $\{y_n\} \subset L(A,\tau) \subset L(M,\tau)_{SA}$ the following conditions are equivalent.

(i) $\{y_n\}$ converges a.u. in $L(M,\tau)$;

(ii) Given $\varepsilon > 0$, there is $e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ s.t. $\|(y_m - y_n)e\| \to 0 \ as \ m,n \to \infty;$

(iii) $\{y_n\}$ converges b.s.a.u.in $L(A,\tau);$

(iv) Given $\varepsilon > 0$, there is $e \in P(M)$ with $\tau(e^{\perp}) < \varepsilon$ s.t. $\|e(y_m - y_n)^2 e\| \to 0 \ as \ m,n \to \infty;$

Proof: conditions (i) $\Rightarrow (ii) \Rightarrow (iii)$ are trivial. For (iv) : From

$$\|e(y_m - y_n)^2 e\| = \|((y_m - y_n)e)^*(y_m - y_n\|$$
$$\le \|((y_m - y_n)e)^*\| . \|(y_m - y_n)e\|$$
$$= \|(y_m - y_n)e\|^2,$$

so we can see that b.s.a.u. fundamentalness of a sequence in a reversible JW-algebra is equivalent to a.u. fundamentalness of the same sequence in its von Neumann enveloping algebra M = M(A). Thus the statement follows from proposition 1 above and hence the proved.

The Riesz theorem above will now take the following form

Theorem 3: If $\{y_n\} \subset L(A,\tau)$ and $y = t_\tau - \lim_{n\to\infty} y_n$, then $y = b.s.a.u. - \lim_{k\to\infty} y_{n_k}$ for some subsequence $\{y_{n_k}\} \subset \{y_n\}$.

Proof: Directly from propositions 1 and 2 respectively.

3. Uniform Equicontinuity for Sequences of Maps into $L(M,\tau)$

Let E be any set. If $a_n : E \to L, x \in E,$ and $b \in M$ are such that $\{a_n(x)b\} \subset M$, then we denote

$$S(x,b) = S(\{a_n\}, x, b) = \sup_n \|a_n(x)b\|.$$

The following fact should be noted.

Corollary: Let $(X,+)$ be a semigroup, $a_n : X \to L$ be sequence of additive maps. Assume that $\bar{x} \in X$ is such that for every $\varepsilon > 0$ there exist a sequence $\{x_k\} \subset X$ and a projection $p \in P(M)$ with $\tau(e^\perp) \le \varepsilon$ such that

(i) $\{a_n(x + x_k)\}$ converges a.u. as $n \to \infty$ for every *k*;

(ii) $S(s_k, p) \to 0, k \to \infty.$

Then the sequence $\{a_n(x)\}$ converges a.u. in L.

Proof: Fix $\varepsilon > 0,$ and let $\{x_k\} \subset X$ and $p \in P(M)$ with $\tau(p^\perp) \le \frac{\varepsilon}{2}$ be such that the two conditions hold. Now pick $\delta > 0$ and let $k_0 = k_0(\delta)$ be such that $S(x_{k_0}, p) \le \frac{\varepsilon}{3}.$ then by proposition 1, there is a projection $q \in P(M)$ with $\tau(q^\perp) \le \frac{\varepsilon}{2}$ and a positive integer N for which the inequality $\|(a_m(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))q\| \le \frac{\delta}{3}$ holds whenever $m, n \ge N$. If we define $e = p \wedge q$, then $\tau(e^\perp) = \tau(p \wedge q)^\perp = \tau(p^\perp) + \tau(q^\perp) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon$ and $\|(a_m(\bar{x}) - a_n(\bar{x}))e\| \le \|(a_m(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))e\|$ $+ \|a_m(x_{k_0})e\| + \|a_n(x_{k_0})e\| \le \delta$ for all $m, n \ge N$. Therefore by proposition 1, the sequence $\{a_n(\bar{x})\}$ converges a.u. in L.

The following definitions hold:

Definition: A sequence $\{a_n\}$ is said to be equicontinuous at x_0 if, given $\varepsilon > 0$ *and* $\delta > 0$, there is a neighborhood U of x_0 in (X,t) such that $a_n U \subset V(\varepsilon, \delta), n = 1, 2, 3, ...$ i.e. for every $x \in U$ and every n one can find a projection $e = e(x,n) \in P(M)$ with $\tau(e^\perp) \le \varepsilon$ satisfying $\|a_n(x)e\| \le \delta.$

Now let (X,t) be a topological space, and let $a_n : X \to L$ and $x_0 \in X$ be such that $a_n(x_0) = 0,\ n = 1, 2, ...$

Definition: Let $(X,t),\ a_n : X \to L,$ and $x_0 \in X$ be as above. Let $x_0 \in E \subset X.$ The sequence $\{a_n\}$ will be called *uniformly equicontinuous* at x_0 on E if, given $\varepsilon > 0, \delta > 0$, there is a neighborhood U of x_0 in (X,t) such that for every $x \in E \cap U$ there exists a projection $e = e(x) \in P(M), \tau(e^\perp) \le \varepsilon,$ satisfying $S(x,e) \le \delta.$

4. Bilateral with Square Uniform Equicontinuity for Sequences of Maps into $L(A,\tau)$

Definition: Let (X,t) be a topological space, $a_n : X \to L(A,\tau)$ and $x_0 \in X$ be such that $a_n(x_0) = 0$ for $n \in N$. A sequence $\{a_n\}$ is called bilaterally with square equicontinuous at x_0 in (X,t) such that $a_n U \subset V(\varepsilon,\delta) \cap L(A,\tau),\ n \in N,$ i.e. for all $x \in U$ *and* for all $n \in N$ one can find a projection $e = e(x,n) \in P(A)$ with $\tau(e^\perp) < \varepsilon,$ satisfying $\|e(a_n(x)^2 e\| < \delta$.

Definition: Let $x_0 \in E \subset X.$ A sequence $\{a_n\}$ is called bilaterally with square uniformly equicontinuous at x_0 on E, if for all $\varepsilon, \delta > 0,$ there is a neighborhood U of x_0 in (X,t) such that $a_n U \subset V(\varepsilon,\delta) \cap L(A,\tau),\ n \in N,$ i.e. for all $x \in U$ and for all $n \in N$ one can find a projection $e = e(x,n) \in P(A)$ with $\tau(e^\perp) < \varepsilon,$ satisfying $\|e(a_n(x))^2 e\| < \delta.$

We can now define a bilaterally sequence with square uniformly equicontinuous as follows:

Definition: Let $x_0 \in E \subset X$. A sequence $\{a_n\}$ is called *bilaterally with square uniformly equicontinuous* at x_0 on E, if for all $\varepsilon, \delta > 0$ there is a neighborhood U of

x_0 in (X,t) such that for all $x \in E \cap U$, there is $e = e(x) \in P(A)$ with $\tau(e^{\perp}) < \varepsilon$, satisfying $S(\{a_n^2\}, x, e) < \delta$.

We then have the following result

Proposition 3: Let the sequence $\{a_n\}$ and $x_0 \in E \subset X$ as in definition above. Then,

(i) $\{a_n\}$ is equicontinuous at x_0 on E into $L(M,\tau)$ iff it is bilaterally with square equicontinuous at x_0 on E into $L(A,\tau)$;

(ii) $\{a_n\}$ is uniformly equicontinuous at x_0 on E into $L(M,\tau)$ iff it is bilaterally with square uniformly equicontinuous at x_0 on E into $L(A,\tau)$.

Proof: Directly follows from proposition 2 and arguments in [7]

Also in [7] it has been established that for any $d > 0$, the sets

$$M_d = \{x \in M : \|x\| \le d\}, \text{ and}$$

$$M_d^h = \{x \in M_d : x = x^*\} \text{ are } t_\tau - \text{complete.}$$

Therefore, $(L(M,\tau), t_\tau)$ is a complete metrizable topological $*-$algebra, and $(L(A,\tau), t_\tau)$ is a complete metrizable topological Jordan subalgebra of $(L(M,\tau), t_\tau)_{SA}$. Hence, it is easy to see that the set $A_d = M_d^h \cap A$ is $t_\tau -$complete as well.

5. Main Results

In the $*-$algebra, let's consider the following conditions. Let $0 \in E \subset M$, now for a sequence of function $a_n : (M, t_\tau) \to L$, then

(i) almost uniform convergence of $\{a_n(x)\}$ for every $x \in E$; $(CNV(E))$

(ii) uniform equicontinuity at 0 on E; $(CNT(E))$

(iii) closedness in (E, t_τ) of the set

$$C(E) = \{x \in E : \{a_n(x)\}\, converges\, a.u.\}$$
$$(CLS(E))$$

With these conditions, one can study relationships among the conditions $(CNV(M_1))$, $(CNT(M_1))$, and $(CLS(M_1))$. Following the classical scheme, one more condition can be added, namely, a non-commutative version of the existence of the maximal operator as follows: given $x \in E$ and $\varepsilon > 0$ there is $e \in P(M), \tau(e^{\perp}) \le \varepsilon$, with $S(x,e) < \infty$. $(BND(E))$. This condition may be called a *pointwise uniform boundedness* of $\{a_n\}$ on E. It can be easily verified that $(CNV(E))$ implies $(BND(E))$, but $(BND(M_1))$ does not guarantee $(CNT(M_1))$. However, if a_n is additive for every n, then $(CNV(M))$ follows from $(CNV(M_1))$ while if E is closed in (M, t_τ), then $(CLS(E))$ is equivalent to the closedness of $C(E)$ in (L, t_τ).

In the JW-algebras, the above conditions can also be extended as follows: Let $0 \in E \subset A$. For a sequence $a_n : (A, t_\tau) \to L(A,\tau)$, then

(i) Bilateral with square almost uniform convergence of $\{a_n(x)\}$ for every $x \in E$ $(BSCNV(E))$;

(ii) Bilateral with square uniform equicontinuity at 0 on E $(BSCNT(E))$;

(iii) Closedness in (E, t_τ) of the set

$$C(E) = \{x \in E : \{a_n(x)\}\, converges\, b.s.a.u.\}$$
$$(BSCLS(E))$$

We can then discuss relationships among the conditions $(BSCNV(A_1)), (BSCNT(A_1))$, and $(BSCLS(A_1))$ as discussed in the case of the *-algebras. These are summarized below in the following theorems and whose proofs are obtained directly in [7] and the arguments in this paper.

Theorem 4: Let $a_n : A \to L(A,\tau)$ be a $(BSCNV(A_1))$ sequence of positive $t_\tau -$ continuous linear maps with $a_n(1) \le 1, n \in N$. Then the sequence $\{a_n\}$ is also $(BSCNT(A_1))$.

Theorem 5: A $(BSCNT(A_1))$ sequence of additive maps $a_n : A \to L(A,\tau)$ is as well $(BSCLS(A_1))$.

Theorem 6: Let $a_n : A \to L(A,\tau)$ be a sequence of positive $t_\tau -$ continuous linear maps such that $a_n(1) \le 1, n \in N$. If a sequence $\{a_n\}$ is $(BSCNV(D))$ $with\ D$ being $t_\tau - dense\ in\ A_1$, the conditions $(BSCNV(A_1)), (BSCNT(A_1)),$ and $(BSCLS(A_1))$ are equivalent.

6. Conclusions

The results in this paper has shown how Banach Principle for semifinite Von Neumann (W^*-algebras) algebras was extended to the case of $JW - a\lg ebras$ without direct summand of type I_2. We can extend these results to the case of bilateral almost uniform convergence on semifinite von

Neumann algebras and semifinite $JBW-a\lg ebras$ without direct summand of type I_2. These results can further be extended to obtain Stochastic Banach Principle, and then apply it to obtain some new Ergodic type theorems for Jordan algebras.

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