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Mathematical Analysis of Convective Flow of an Unsteady Magnetohydrodynamic (MHD) Third Grade Fluid in a Cylindrical Channel

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Abstract This paper presents magnetohydrodynamic (MHD) flow of unsteady convective third grade fluid in a cylindrical system. The non-linear governing equations were solved analytically using homotopy perturbation method (HPM). The influences of dimensionless parameters on magnetohydrodynamic (MHD) flow of a convective third grade fluid in a cylindrical system were investigated. Simulation results revealed that temperature and the velocity depend on the values of combination of magnetic field and porosity. The obtained results are presented graphically and discussed. It is observed that velocity decreases and increases with increasing magnetic field and porosity, temperature increases as magnetic field increases.

Keywords Magnetohydrodynamic, Perturbation, Convective Third grade fluid

1. Introduction

In most recent years, authors carried out investigation and problem dealing with the flow of non-Newtonian fluid in a cylindrical system. This interest is due to several important applications in engineering and industry such as reactive polymer flows in heterogeneous porous media, extraction of crude oil from the petroleum production, synthetic fibres and paper production (Schowaller 1978). Many practical situations we have deals with the natural convection of heat which play significant role in the behaviour of the flow.

Convection problems associate will heat sources within fluid saturated porous media are of great practical signification, such as in geophysics and energy related problems (petroleum resources, geophysical flows, cooling of underground electric cable e.t.c). In generally terms, the difference between Non-Newtonian fluids and the single component Newtonian fluid in that, in the latter case the mathematical formulation is known but the macroscopic physical processes are complex and often not well understood, especially for turbulent flow conditions are for Non - Newtonian fluids even the appropriation governing equations and conditions at the boundaries are still not well understood. However, the flow of Non - Newtonian fluids plays an important role in many practical applications.

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Sajid et al (2008) discussed the effect of variable viscosity on the flow and heat transfer in a thin film flow for a third grade fluid. The thin film was considered on the outer side of an infinitely long vertical cylinder. The governing non-linear differential equation of momentum and energy were solved analytically by using homotopy analysis method (HAM). The study of MHD convection in a vertical channel, Aiyesimi et al. (2013) considered the MHD heat transfer in the flow between 2 concentric cylinders.

Siddiqui et al (2012) carried out studies on two phase flow of a third grade fluid between parallel plates in three different cases. Makinde et al. (2010) studied MHD mixed convection from a vertical plate embedded in a porous medium with a convective boundary condition. Chamkha and Ahmad (2012) investigated unsteady MHD heat and mass transfer by mixed convection flow in the forward stagnation region of a rotating sphere at different wall conditions. Unsteady MHD convective heat and mass transfer in a boundary layer slip flow part a vertical permeable with thermal radiation and chemical reaction was examined by Dulal and Talukdar (2000). Singh and Pathak (2010) studied effect of slip condition on rotating vertical channel. Kandasamy et al. (2011) group theory transformation for Soret and Dufour effects on free convective heat and mass transfer with thermophoresis and chemical reaction over a porous stretching surface in the presence of heat source/sink. Rao *et al*. (2012) have found out the chemical effects on an unsteady MHD free convection fluid past a semi-infinite vertical plate embedded in a porous medium with heat absorption.

In this paper, magnetohydrodynamics (MHD) flows of an unsteady convective third grade fluid in a cylindrical system were investigated. The governing equations arising from the unsteady flow are solved using homotopy perturbation method (HPM).

2. Model Formulation

The one-dimensional momentum and energy equations describing MHD flows of an unsteady convective third grade fluid in a cylindrical system are:

Momentum equation

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)=\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-p+2\alpha_1\left(\frac{\partial u}{\partial r}\right)\right)+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+6\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)+2\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial r}+\alpha_1\frac{\partial^2 u}{\partial t\partial r}+\beta_1\frac{\partial^3 u}{\partial t^2\partial r}+2\beta_2\left(\frac{\partial u}{\partial r}\right)^3+2\beta_3\left(\frac{\partial u}{\partial r}\right)^3\right)-\sigma\beta_0^2 u-\mu\frac{\emptyset}{k}u \quad (1)$$

Energy equation

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}\right)=k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+2\mu\left(-p+2\alpha_1\left(\frac{\partial u}{\partial r}\right)^2+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+6\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)+2\beta_2\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)\right)^2+\sigma\beta_0^2 u^2 \quad (2)$$

together with initial and boundary conditions

$$u(r,0)=U\left(1-\frac{r}{R}\right),\qquad u(0,t)=U,\qquad \left.\frac{\partial u}{\partial r}\right|_{r=R}=0$$

$$T(r,0)=(T_1-T_0)\left(1-\frac{r}{R}\right)+T_0,\ T(0,t)=T_0,\ T(R,t)=T_1 \quad (3)$$

where u is velocity, p is pressure, k is the thermal conductivity, T is the temperature, $\rho\ is\ the$ fluid c_p specific heat, μ fluid viscosity, B_0 applied magnetic field, t is the non-dimensional time, σ electrical conductivity.

Here, equations (1) – (3) are transformed using the following coordinate transformations:

$$\frac{\partial}{\partial r}\longrightarrow\frac{\partial}{\partial\eta}\cdot\frac{\partial\eta}{\partial r}=\rho\frac{\partial}{\partial\eta} \quad (4)$$

$$\frac{\partial}{\partial t}\longrightarrow\frac{\partial}{\partial\eta}\cdot\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial\eta}=-\rho u\frac{\partial}{\partial\eta}+\frac{\partial}{\partial t} \quad (5)$$

$$r=\frac{\eta}{\rho} \quad (6)$$

and we obtain

$$\rho\frac{\partial u}{\partial t}=\rho\frac{\partial p}{\partial\eta}+\frac{\rho}{\eta}\left(-p+2\alpha_1\rho^2\frac{\partial^2 u}{\partial\eta^2}+2\alpha_2\rho^3\frac{\partial^2 u}{\partial\eta^2}\frac{\partial u}{\partial\eta}+(6\beta_1+2\beta_2)\rho^3\left(\frac{\partial^2 u}{\partial t\partial\eta}\cdot\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}\cdot\frac{\partial^3 u}{\partial t\partial\eta^2}\right)\right)$$

$$+\frac{\rho}{\eta}\left(\mu\rho^2\frac{\partial^2 u}{\partial\eta^2}+\alpha_1\rho^2\frac{\partial^3 u}{\partial t\partial\eta^2}+\beta_1\rho^2\frac{\partial^4 u}{\partial t^2\partial\eta^2}+6(\beta_2+\beta_3)\rho^2\frac{\partial^2 u}{\partial\eta^2}\cdot\left(\rho\frac{\partial u}{\partial\eta}\right)^2\right)-\sigma B_0^2 u-\mu\frac{\emptyset}{k}u \quad (7)$$

$$\rho c_p\frac{\partial T}{\partial t}=k\rho^2\frac{\partial^2 T}{\partial\eta^2}+k\frac{\rho^2}{\eta}\frac{\partial T}{\partial\eta}+2\mu\left(-p+(2\alpha_1+\alpha_2)\rho^2\left(\frac{\partial u}{\partial\eta}\right)^2+(6\beta_1+2\beta_2)\rho^2\frac{\partial u}{\partial\eta}\frac{\partial^3 u}{\partial t\partial\eta^2}\right)^2+\sigma B_0^2 u^2 \quad (8)$$

together with initial and boundary conditions;

$$u(\eta,0)=U\left(1-\frac{\eta}{R\rho}\right),\qquad u(0,t)=U,\qquad \left.\frac{\partial u}{\partial\eta}\right|_{\eta=R}=0$$

$$T(\eta,0)=(T_1-T_0)\left(1-\frac{\eta}{R\rho}\right)+T_0,\ T(0,t)=T_0,\ T(R,t)=T_1 \quad (9)$$

3. Method of Solution

3.1. Non-dimensionalization

Here, we let pressure p be constant and non-dimensionlised equations (7) – (9) using the following dimensionless variables,

$$\eta'=\frac{\eta}{R},\qquad u'=\frac{u}{U},\qquad t'=\frac{tU}{R},\qquad \theta=\frac{T-T_0}{T_1-T_0} \quad (10)$$

and obtain

$$\frac{\partial u}{\partial t}=\frac{\alpha}{\eta}+\frac{\beta}{\eta}\frac{\partial^2 u}{\partial\eta^2}+\frac{\sigma_1}{\eta}\frac{\partial^2 u}{\partial\eta^2}\left(\frac{\partial u}{\partial\eta}\right)+\frac{\sigma_2}{\eta}\left(\frac{\partial^2 u}{\partial t\partial\eta}.\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}.\frac{\partial^3 u}{\partial t\partial\eta^2}\right)+\frac{\sigma_3}{\eta}\frac{\partial^2 u}{\partial\eta^2}+\frac{\sigma_4}{\eta}\frac{\partial^3 u}{\partial t\partial\eta^2}+\frac{\sigma_5}{\eta}\frac{\partial^4 u}{\partial t^2\partial\eta^2}+\frac{\sigma_6}{\eta}\frac{\partial^2 u}{\partial\eta^2}\left(\frac{\partial u}{\partial\eta}\right)^2-\sigma_7 u \quad (11)$$

$$\frac{\partial\theta}{\partial t} = Pr\frac{\partial^2\theta}{\partial\eta^2} + \frac{Re}{\eta}\frac{\partial\theta}{\partial\eta} + \left(-2Br + 2Gr\gamma\left(\frac{\partial u}{\partial\eta}\right)^2 + 2Ec\frac{\partial u}{\partial\eta}\left(\frac{\partial^2 u}{\partial t\partial\eta}\right)\right)^2 + Mu^2 \tag{12}$$

together with initial and boundary conditions:

$$u(\eta,0) = \left(1-\frac{\eta}{\rho}\right),\ \ u(0,t) = 1,\qquad \frac{\partial u}{\partial\eta}\bigg|_{\eta=R} = 0$$

$$\theta(\eta,0) = 1-\frac{\eta}{\rho},\ \ \theta(0,t) = 0,\ \ \theta(1,t) = 1 \tag{13}$$

where

$$\alpha = \frac{-p}{U^2},\quad \beta = \frac{2\alpha_1\rho}{U},\qquad \sigma_1 = \frac{2\alpha_2\rho^2}{R},\qquad \sigma_2 = (6\beta_1 + 2\beta_2)\frac{\rho^3 U}{R^2}$$

$$\sigma_3 = \frac{\rho^2\mu}{UR},\ \ \sigma_4 = \frac{\alpha_1\rho^2}{R},\qquad \sigma_5 = \frac{\beta_1\rho^2}{R},\qquad \sigma_6 = 6(\beta_2+\beta_3)\frac{\rho^4}{R^2}U$$

$$\sigma_7 = \frac{R}{\rho U}\left(\sigma B_0^2 - \mu\frac{\emptyset}{k}\right),\qquad Ec = \frac{\mu\rho U^2}{R^2 c_p(T_1-T_0)},\ Br = \frac{\mu p U}{U\rho c_p(T_1-T_0)},\ Re = \frac{K\rho}{UR},$$

$$Gr = \frac{\mu\rho U}{Rc_p(T_1-T_0)},\quad Pr = \frac{K\rho}{Uc_p},\quad \gamma = (2\alpha_1+\alpha_2),\quad M = \frac{\sigma U B_0^2}{\rho c_p(T_1-T_0)}$$

3.2. Existence and Uniqueness of Solution

Theorem 3.1: Let $Pr = \beta + \sigma_3$, $Re = Br = Gr = Ec = 0, \alpha = \sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$. Then, the equations (11) and (12) with initial and boundary conditions (13) has a unique solution for all $t \geq 0$.

Proof; Let $Pr = \beta + \sigma_3$, $Re = Br = Gr = Ec = 0, \alpha = \sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$. In equations (11) and (12), we obtain

$$\frac{\partial u}{\partial t} = \frac{Pr}{\eta}\frac{\partial^2 u}{\partial\eta^2} - \sigma_7 u \tag{14}$$

$$u(\eta,0) = \left(1-\frac{\eta}{\rho}\right),\ \ u(0\,,t) = 1,\ \ u_\eta(1,t) = 0$$

$$\frac{\partial\theta}{\partial t} = Pr\frac{\partial^2\theta}{\partial\eta^2} + Mu^2 \tag{15}$$

$$\theta(\eta\,,0) = \left(1-\frac{\eta}{\rho}\right),\quad \theta(0,t) = 0,\quad \theta(1,t) = 1$$

Using frobenius method, we obtain the solution of problem (14) as

$$u(\eta,t) = \left(\left(1+\eta+\frac{\sigma_7}{6}\eta^3+\frac{\sigma_7}{12}\eta^4+\cdots\right) + \left(\frac{1+\frac{\sigma_7}{2}\eta^2+\frac{\sigma_7}{3}\eta^3+\cdots}{1+\frac{\sigma_7}{3}\eta^3+\cdots}\right)\left(\eta+\frac{\sigma_7}{12}\eta^4+\cdots\right)\right) \tag{16}$$

where

$$u_n(t) = \frac{4}{(2n-1)\pi}\left(1+\frac{(-1)^n}{(2n-1)\pi}\right)e^{-\left(\sigma_7+Pr\left(\frac{2n-1}{2}\pi\right)^2\right)t}$$

and using eigenfunction expansion method, the solution of problem as

$$\theta(\eta,t) = \eta + \textstyle\sum_{n=1}^{\infty} V_n\,(t)\sin n\pi\eta, \tag{17}$$

where

$$V_n(t) = q_n(t) + b_n e^{-Pr(n^2\pi^2)t}$$

$$q_n(t) = 2M\int_0^t T_n(\tau)e^{-Prn^2\pi^2(t-\tau)}$$

$$T_n(\tau) = \int_0^1\left(\sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{2n-1}{2}\right)\eta\pi\right)^2 \sin n\pi\eta d\eta$$

Hence, there exists a unique solution of problems (11) and (12). This completes the proof.

3.3. Analytical Solution

Applying homotopy perturbation method (HPM), we have

$$(1-p)\frac{\partial u}{\partial t}+p\begin{pmatrix}\frac{\partial u}{\partial t}-\frac{\alpha}{\eta}-\frac{\beta}{\eta}\left(\frac{\partial^2 u}{\partial \eta^2}\right)-\frac{\sigma_1}{\eta}\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial u}{\partial \eta}\right)-\frac{\sigma_2}{\eta}\left(\frac{\partial^2 u}{\partial t\partial \eta}\cdot\frac{\partial^2 u}{\partial \eta^2}\right)-\frac{\sigma_2}{\eta}\left(\frac{\partial u}{\partial \eta}\cdot\frac{\partial^3 u}{\partial t\partial \eta^2}\right)\\ -\frac{\sigma_3}{\eta}\left(\frac{\partial^2 u}{\partial \eta^2}\right)-\frac{\sigma_4}{\eta}\left(\frac{\partial^3 u}{\partial t\partial \eta^2}\right)-\frac{\sigma_5}{\eta}\left(\frac{\partial^4 u}{\partial t^2\partial \eta^2}\right)-\frac{\sigma_6}{\eta}\left(\frac{\partial^2 u}{\partial \eta^2}\right)\left(\frac{\partial u}{\partial \eta}\right)^2+\sigma_7 u\end{pmatrix}=0 \tag{18}$$

$$(1-p)\frac{\partial \theta}{\partial t}+p\left(\frac{\partial \theta}{\partial t}-Pr\left(\frac{\partial^2 \theta}{\partial \eta^2}\right)-\frac{Re}{\eta}\left(\frac{\partial \theta}{\partial \eta}\right)-\left(-2Br+2Gr\gamma\left(\frac{\partial u}{\partial \eta}\right)^2+2Ec\frac{\partial u}{\partial \eta}\left(\frac{\partial^2 u}{\partial t\partial \eta}\right)\right)^2-Mu^2\right)=0 \tag{19}$$

with initial and boundary condition

$$u(\eta,0)=\left(1-\frac{\eta}{\rho}\right),\quad u(0,t)=1,\qquad \left.\frac{\partial u}{\partial \eta}\right|_{\eta=R}=0$$
$$\theta(\eta,0)=1-\frac{\eta}{\rho},\qquad \theta(0,t)=0,\qquad \theta(1,t)=1 \tag{20}$$

Therefore, we obtain the following at zero order:
Momentum equation

$$u_0(\eta,t)=\left(1-\frac{\eta}{\rho}\right)$$

Energy equation

$$\theta_0(\eta,t)=\left(1-\frac{\eta}{\rho}\right)$$

at first - order Problem, we have
Momentum equation:

$$\frac{\partial u_1}{\partial t}-\frac{\alpha}{\eta}+\sigma_7\left(1-\frac{\eta}{\rho}\right)=0$$

$$u_1(\eta,t)=\left(\frac{\alpha}{\eta}-\sigma_7\left(1-\frac{\eta}{\rho}\right)\right)t-\sigma_7$$

Energy equation:

$$\theta_1(\eta,t)=\left(\left(-2Br+\frac{2Gr\gamma}{\rho}\right)^2+M\left(1-2\eta+\left(\frac{\eta}{\rho}\right)^2\right)-\frac{Re}{\rho\eta}\right)t-\left(\left(-2Br+\frac{2Gr\gamma}{\rho}\right)^2\right)t$$

we let

$$u=u_0+p^1u_1+p^2u_2+\cdots$$
$$\theta=\theta_0+p^1\theta_1+p^2\theta_2+\cdots$$
$$u(\eta,t)=\left(1-\frac{\eta}{\rho}\right)+p\left(\frac{\alpha}{\eta}-\sigma_7\left(1-\frac{\eta}{\rho}\right)\right)t-\sigma_7$$
$$\theta(\eta,t)=\left(1-\frac{\eta}{\rho}\right)+p\left(\left(-2Br+\frac{2Gr\gamma}{\rho}\right)^2+M\left(1-2\eta+\left(\frac{\eta}{\rho}\right)^2\right)-\frac{Re}{\rho\eta}\right)t$$

The computations were done using algebraic package MAPLE.

4. Results and Discussion

The analytical results obtained are shown through graphs which demonstrate the various parameter on the velocity and temperature of magnetohyrodynamic (MHD) flow of a convective third grade fluid in a cylindrical system by using homotopy perturbation method (HPM). We prove the existence and unigueness of solution by the actural method.

5. Discussion

The effect of σ_7 (combination of magnetic field and porosity) on velocity (u) for $\alpha=1,\ \rho=1,\ t=5$ is shown in fig.1. It is observed that, velocity (u) decreases with combination of magnetic field and porosity increases. Fig.2, Shows the effect of σ_7 (combination of magnetic field and porosity) on velocity (u) for $\alpha=1,\ \rho=1,\ \eta=0.5$. We noted that, the velocity (u) increases with magnetic field and porosity increases.

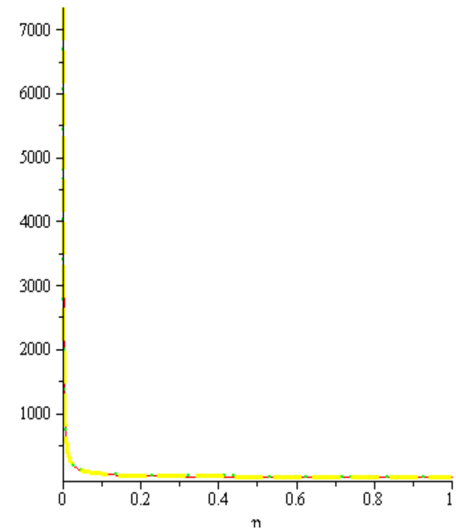

Figure 1.

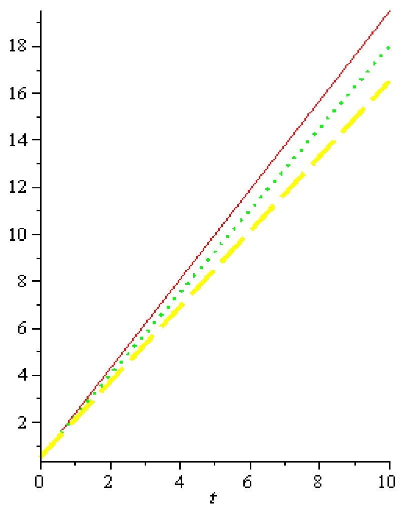

Figure 2.

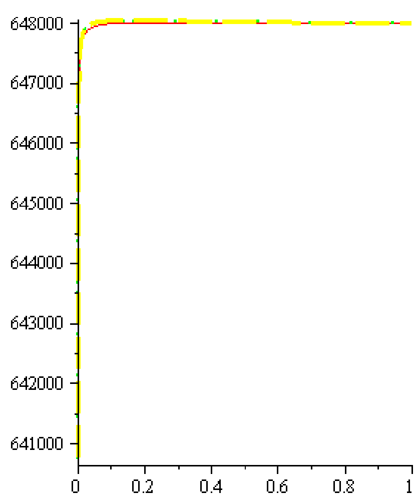

Figure 3.

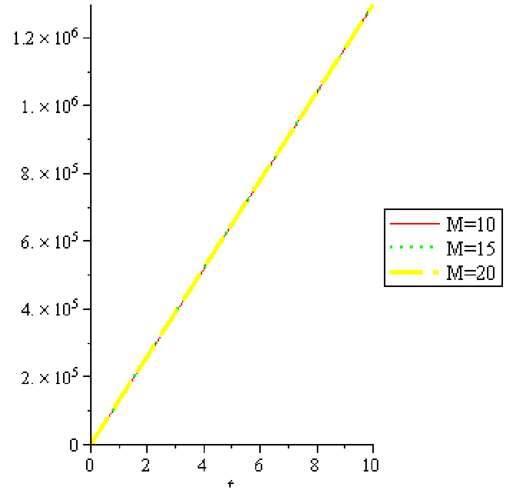

Figure 4.

The effect of magnetic field (M) on temperature (T) for $Br = 15,\ Gr = 15,\ \gamma = 13,\ Re = 1,\ t = 5$ is shown in fig.3, it is observed that temperature (T) increases with magnetic field increases. It is also found that, in fig. 4 as temperature (T) increases, magnetic field increases for $Br = 15,\ Gr = 15,\ \gamma = 13,\ Re = 1,\ \eta = 0.5$.

6. Conclusions

The MHD flows of unsteady convective third grade fluid in cylindrical systems were examined. The resulting equations were solved using HPM and graphical results were obtained. The unsteady flow study were analysed through the effects of physical parameters such as Magnetic field on velocity and temperature distribution.

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