American Journal of Computational and Applied Mathematics 2016, 6(2): 65-73
DOI: 10.5923/j.ajcam.20160602.06

# Dynamic Output Feedback Controller Design for a Class of Takagi-Sugeno Descriptor Systems

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**Abstract** This paper considers the design problem of dynamic output feedback controller for a class of nonlinear descriptor systems described by Takagi-Sugeno (T-S) fuzzy models with measurable premise variables. The approach is based on the separation between dynamic and static relations in the descriptor model. The convergence of the closed-loop state system is studied by using the Lyapunov theory and the stability conditions are given in terms of linear matrix inequalities (LMIs). In practice, the computation of solutions of descriptor systems requires the combination of an ordinary differential equation (ODE) routine together with an optimization algorithm. The main result of this paper consists in showing that the dynamic output feedback controller problem for a class of T-S descriptor systems can be achieved by using a fuzzy controller based on an ODE structure only. Finally, simulation results are provided in order to show the validity of the proposed method.

**Keywords** Fuzzy dynamic output feedback controller, Takagi-Sugeno descriptor systems, Measurable premise variables, Linear matrix inequality (LMI)

## 1. Introduction

In practice, in order to realize a feedback stabilization, we must know the state of the system. This state can be obtained by several ways. One way consists to design an observer. The theory which combines the couple (observer, controller system) is variously called separation principle, dynamic output feedback controller or observer-based controller. In this paper, we are concerned with the separation approach to the design of stabilizing output feedback control using T-S fuzzy observer.

Recall that, T-S models [1] have been widely used for analysis and controller synthesis of nonlinear systems. The advantage of such approach relies on the fact that once the T-S fuzzy models are obtained, some analysis and design tools developed in the linear case can be used, which facilitates observer or/and controller synthesis for complex nonlinear systems see for example [2] and the references therein. However, many research on T-S fuzzy control and state observation for nonlinear systems described by ordinary differential equations (ODEs) exist in the literature. Indeed, the stability and the stabilization of such systems are mainly investigated through the direct Lyapunov method [3], [4]. Likewise, various works have been proposed to design observers [5-8]. The observer-based controller synthesis has been studied extensively with great success for such nonlinear systems. We may cite [9-13]. For nonlinear descriptor systems described by T-S models, the problems of control and observation design are mainly investigated in the literature, see for instance [14-17]. Notice that, generally an interesting way to solve the various problems raised previously (control and observation) is to write the convergence conditions on the LMI form.

In this paper, we deal with descriptor nonlinear systems. In fact, many chemical and physical processes can be described by nonlinear systems of differential and algebraic equations [18-20]. These systems are variously called descriptor systems, singular systems, or differential algebraic equations (DAEs). This formulation includes both dynamic and static relations. The numerical simulation of such descriptor models usually combines an ODE numerical method together with an optimization algorithm.

The main contribution of this paper is to give an observer-based controller design for a class of T-S descriptor models with measurable premise variables. A new design methodology through judicious use of the inequality of Young is proposed. The developed result is based on the separation between the dynamic and static relations in the descriptor model. The asymptotic stability of the closed-loop state system is studied by using the Lyapunov theory and the stability conditions are given in terms of LMIs. Besides, the proposed dynamic controller for a class of T-S descriptor systems can be synthesized by only an ODE structure.

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Published online at http://journal.sapub.org/ajcam

The outline of the paper is as follows. The class of the fuzzy T-S structure of nonlinear descriptor systems is introduced in Section 2. The main result about fuzzy observer-based controller design for a class of T-S descriptor models with measurable premise variables is stated in sections 3. The control and observer gains are found directly from LMI formulation. An illustrative application is given in Section 4 to show the efficiency of the proposed approach.

In this paper, some notations used are fair standard. For example, $X>0$ means the matrix $X$ is symmetric and positive definite. $X^T$ denotes the transpose of $X$. The symbol $I$ (or $0$) represents the identity matrix (or zero matrix) with appropriate dimension, and $\sum_{i,j=1}^{q}\mu_i\mu_j=\sum_{i=1}^{q}\sum_{j=1}^{q}\mu_i\mu_j$.

## 2. System Description

In this paper, the following class of T-S descriptor systems is considered:

$$\begin{cases} E\dot{x}=\sum_{i=1}^{q}\mu_i(\xi)(A_ix+B_iu) \\ y=\sum_{i=1}^{q}\mu_i(\xi)C_ix \end{cases} \tag{1}$$

Where $x=[X_1^T\,X_2^T]^T\in\mathbb{R}^n$ is the state vector with $X_1\in\mathbb{R}^r$ is the vector of differential variables, $X_2\in\mathbb{R}^{n-r}$ is the vector of algebraic variables, $u\in\mathbb{R}^m$ is the control input, $y\in\mathbb{R}^p$ is the measured output. $E\in\mathbb{R}^{n\times n}$ with $rank(E)=r$, $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times m}$, $C_i\in\mathbb{R}^{p\times n}$ are real known constant matrices with:

$$\begin{aligned} E&=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix};\quad A_i=\begin{pmatrix} A_{11i} & A_{12i}\\ A_{21i} & A_{22i}\end{pmatrix};\\ B_i&=\begin{pmatrix} B_{1i}\\ B_{2i}\end{pmatrix};\quad C_i=\begin{pmatrix} C_{1i} & C_{2i}\end{pmatrix}\end{aligned} \tag{2}$$

where constant matrices $A_{22i}$ are supposed invertible

($rank(A_{22i})=n-r$).

The premise variable $\xi$ is supposed to be real-time accessible and depended only on $X_1$. $q$ is the number of sub-models. The $\mu_i(\xi)$ ($i=1,\dots,q$) are the weighting functions that ensure the transition between the contribution of each sub model:

$$\begin{cases} E\dot{x}=A_ix+B_iu \\ y=C_ix \end{cases} \tag{3}$$

They verify the so-called convex sum properties:

$$\begin{cases} \sum_{i=1}^{q}\mu_i(\xi)=1 \\ 0\le\mu_i(\xi)\le1 \quad i=1,\dots,q \end{cases} \tag{4}$$

In what follows and to prove the convergence of the observer-based controller proposed in this work, we consider the following assumptions [20]:

**Assumption A1:** $(E,A_i)$ is regular, i.e. $det(sE-A_i)\neq0\quad\forall s\in\mathbb{C}$

**Assumption A2:** All sub-models (3) are impulse controllable and stabilisable.

**Assumption A3:** All sub-models (3) are impulse observable and detectable.

The main objective of this work is to design an observer-based controller for system (1) which assures asymptotic convergence to zero of the closed-loop system. The approach is based on the separation between dynamic and static relations in each sub-model (3) and the global fuzzy model is obtained by aggregation of the resulting sub-models. So, using (2), system (3) can be rewritten as follows:

$$\begin{cases} \dot{X}_1=A_{11i}X_1+A_{12i}X_2+B_{1i}u \\ 0=A_{21i}X_1+A_{22i}X_2+B_{2i}u \\ y=C_{1i}X_1+C_{2i}X_2 \end{cases} \tag{5}$$

The form (5) for system (3) is also known as the second equivalent form [20].

From (5) and using the fact that $A_{22i}^{-1}$ exist, the algebraic equations can be solved directly for algebraic variables, to obtain:

$$X_2=-A_{22i}^{-1}A_{21i}X_1-A_{22i}^{-1}B_{2i}u \tag{6}$$

Substitution of the resulting expression for $X_2$ (equation (6) in equation (5) yields the following model:

$$\begin{cases} \dot{X}_1=M_iX_1+N_iu \\ X_2=Q_iX_1+R_iu \\ y=S_iX_1+G_iu \end{cases} \tag{7}$$

Where

$$\begin{cases} M_i = A_{11i} - A_{12i}A_{22i}^{-1}A_{21i} \\ N_i = B_{1i} - A_{12i}A_{22i}^{-1}B_{2i} \\ Q_i = -A_{22i}^{-1}A_{21i} \\ R_i = -A_{22i}^{-1}B_{2i} \\ S_i = C_{1i} - C_{2i}A_{22i}^{-1}A_{21i} \\ G_i = -C_{2i}A_{22i}^{-1}B_{2i} \end{cases} \tag{8}$$

In descriptor form, subsystems (7) take the following equivalent form of sub-models (3)

$$\begin{cases} E\dot{x} = \bar{M}_i x + \bar{N}_i u \\ y = \bar{S}_i x + G_i u \end{cases} \tag{9}$$

Where

$$\bar{M}_i = \begin{pmatrix} M_i & 0 \\ Q & -I \end{pmatrix};\ \bar{N}_i = \begin{pmatrix} N_i \\ R_i \end{pmatrix};\ \bar{S}_i = \begin{pmatrix} S_i & 0 \end{pmatrix} \tag{10}$$

Then, the fuzzy descriptor system (1) can be rewritten in the following equivalent form:

$$\begin{cases} E\dot{x} = \sum_{i=1}^{q} \mu_i(\xi)(\bar{M}_i X_1 + \bar{N}_i u) \\ y = \sum_{i=1}^{q} \mu_i(\xi)(\bar{S}_i X_1 + G_i u) \end{cases} \tag{11}$$

So using (4) and (10), system (11) can be rewritten as follows:

$$\begin{cases} \dot{X}_1 = \sum_{i=1}^{q} \mu_i(\xi)(M_i X_1 + N_i u) \\ X_2 = \sum_{i=1}^{q} \mu_i(\xi)(Q_i X_1 + R_i u) \\ y = \sum_{i=1}^{q} \mu_i(\xi)(S_i X_1 + G_i u) \end{cases} \tag{12}$$

## 3. Main Results

The objective is to design an observer-based controller for the T-S descriptor system (1). Therefore, first a new feedback controller design method is proposed. Secondly, an observer design approach permitting to estimate the unknown state is given.

### 3.1. Feedback Controller Design

In this section, the goal is to determine the gains $K_{i1}$, $K_{i2}$ of the following control law:

$$u(x) = u(X_1, X_2) = -\sum_{i=1}^{q} \mu_i(\xi)(K_{i1}X_1 + K_{i2}X_2) \tag{13}$$

In order that the closed-loop system (1) is asymptotically stable.

Let the local linear feedback controller related with each submodel (7) given by:

$$u(x) = -K_{i1}X_1 - K_{i2}X_2 \tag{14}$$

and the expression of $X_2$ giving in (7) by:

$$X_2 = Q_i X_1 + R_i u \tag{15}$$

So, in order to be able to state the result as LMIs, the key idea consists in substituting (15) in (14). Then, we obtain:

$$u(x) = -K_i X_1 \tag{16}$$

Where

$$K_i = (I + K_{i2}R_i)^{-1}(K_{i1} + K_{i2}Q_i) \tag{17}$$

Thus, the fuzzy state feedback controller (13) can be rewritten in the following equivalent form:

$$u(x) = -\sum_{i=1}^{q} \mu_i(\xi)K_i X_1 = \tilde{u}(X_1) \tag{18}$$

By substituting (18) into (12), the following closed-loop fuzzy system can be represented as:

$$\begin{cases} \dot{X}_1 = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)\tilde{M}_{ij}X_1 \\ X_2 = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)\tilde{Q}_{ij}X_1 \\ y = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)\tilde{S}_{ij}X_1 \end{cases} \tag{19}$$

Where

$$\begin{cases} \tilde{M}_{ij} = M_i - N_i K_j \\ \tilde{Q}_{ij} = Q_i - R_i K_j \\ \tilde{S}_{ij} = S_i - G_i K_j \end{cases} \tag{20}$$

In order to prove that the closed-loop system (19) converges to zero, it suffice to prove that $X_1$ converges toward zero. Then, the following result can be stated.

**Theorem 1.:** The closed-loop system described by (19) is globally exponentially stable if there exist matrices $Y_1 > 0$, $U_i$, $i = 1,\ldots,q$ verifying the following LMIs:

$$\begin{cases} M_iY_1+Y_1^TM_i^T-N_iU_i-U_i^TN_i^T<0 \quad i=1,\dots,q \\ M_iY_1+Y_1^TM_i^T+M_jY_1+Y_1^TM_j^T-N_iU_j \\ \quad -U_j^TN_i^T-N_jU_i-U_i^TN_j^T<0 \\ i<j\ s.t.\ \mu_i\cap\mu_j\neq\varnothing \end{cases} \tag{21}$$

The fuzzy local feedback gains $K_i$, $i=1,\dots,q$ are given by:

$$K_i = U_iY_1^{-1} \tag{22}$$

Note that, to determine the control gains $K_{i1}$ and $K_{i2}$ given in (13), according to the expression of $K_i$ given in (17), it is clear that we have:

$$K_i = \left[K_{i1}\, K_{i2}\right]\theta_i \tag{23}$$

Where

$$\theta_i = \begin{bmatrix} I \\ Q_i - R_iK_i \end{bmatrix} \tag{24}$$

Then, a particular solution of (23) using the pseudo inverse matrix denoted $(.)^+$ is given by:

$$\left[K_{i1}\, K_{i2}\right] = K_i(\theta_i)^+ \tag{25}$$

**Proof:** Let us consider the quadratic Lyapunov function as follows:

$$V(X_1) = X_1^TP_1X_1, \quad P_1 > 0 \tag{26}$$

The time derivative of the Lyapunov function (26) along the trajectories of the system (19) is obtained as:

$$\dot{V}(X_1) = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)X_1^T(\tilde{M}_{ij}^TP_1 + P_1\tilde{M}_{ij})X_1 \tag{27}$$

Therefore, we have the following stability conditions:

$$\begin{cases} \tilde{M}_{ii}^TP_1 + P_1\tilde{M}_{ii} < 0 \qquad i=1,\dots,q \\ (\dfrac{\tilde{M}_{ij}+\tilde{M}_{ji}}{2})^TP_1 + P_1(\dfrac{\tilde{M}_{ij}+\tilde{M}_{ji}}{2}) < 0 \\ i<j\ s.t.\ \mu_i\cap\mu_j\neq\varnothing \end{cases} \tag{28}$$

From (20), inequality (28) becomes:

$$\begin{cases} M_i^TP_1 - K_i^TN_i^TP_1 + P_1M_i - P_1N_iK_i < 0 \quad i=1,\dots,q \\ M_i^TP_1 - K_j^TN_i^TP_1 + M_j^TP_1 - K_i^TN_j^TP_1 \\ \quad + P_1M_i - P_1N_iK_j + P_1M_j - P_1N_jK_i < 0 \\ i<j\ s.t.\ \mu_i\cap\mu_j\neq\varnothing \end{cases} \tag{29}$$

Let $Y_1 = P_1^{-1}$, $U_i = K_iY$. Premultiply and postmultiply (29) by $Y_1$, to obtain the LMI conditions given in (21). From the Lypunov stability theory, if the LMI conditions (21) are satisfied, the closed-loop system (19) is exponentially asymptotically stable. This completes the proof of Theorem 1.

### 3.2. Observer Design

As it was mentioned before, to apply state feedback control it is necessary to know all the states of the system. In real application, not all the states are available to be measured, thus the necessity to use an observer arises. Therefore, in this section, we propose an observer design for a class of T-S descriptor systems (1). Based on the form (12), the proposed observer takes the following form:

$$\begin{cases} \dot{\hat{X}}_1 = \sum_{i=1}^{q}\mu_i(\xi)(M_i\hat{X}_1 + N_iu - L_i(\hat{y}-y)) \\ \hat{X}_2 = \sum_{i=1}^{q}\mu_i(\xi)(Q_i\hat{X}_1 + R_iu) \\ \hat{y} = \sum_{i=1}^{q}\mu_i(\xi)(S_i\hat{X}_1 + G_iu) \end{cases} \tag{30}$$

where $\hat{X}_1$, $\hat{X}_2$ and $\hat{y}$ denote the estimated state vector and output vector respectively. The activation functions $\mu_i(\xi)$ are the same than those used in the T-S model (15). $L_i$ are the gains of observer witch are to determined such that $\hat{x} = [\hat{X}_1^T\ \hat{X}_2^T]^T$ asymptotically converges to $x = [X_1^T\ X_2^T]^T$.

Denoting the state estimation error by:

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \hat{X}_1 - X_1 \\ \hat{X}_2 - X_2 \end{pmatrix} \tag{31}$$

It follows from (12) and (30) that the estimated error equation can be written as:

$$\begin{cases} \dot{e}_1 = \sum_{i,j=1}^{q}\mu_i(\xi)\mu_j(\xi)\Gamma_{ij}e_1 \\ e_2 = \sum_{i=1}^{q}\mu_i(\xi)Q_ie_1 \end{cases} \tag{32}$$

Where

$$\Gamma_{ij} = M_i - L_iS_j \tag{33}$$

To prove the convergence of the estimation error $e$ toward zero, it suffices to prove from (32), that $e_1$

converges toward zero. The result is stated in the following theorem.

**Theorem 2.:** There exists an observer (30) for (1) if there exist matrices $P_2 > 0$, $W_i$, $i = 1, \ldots, q$ verifying the following LMIs:

$$\begin{cases} M_i^T P_2 + P_2 M_i - S_i^T W_i^T - W_i S_i < 0 \quad i = 1, \ldots, q \\ M_i^T P_2 + M_j^T P_2 + P_2 M_i + P_2 M_j - S_j^T W_i^T \\ \qquad - S_i^T W_j^T - W_i S_j - W_j S_i < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \emptyset \end{cases} \tag{34}$$

The fuzzy local observer gains $L_i$, $i = 1, \ldots, q$ are given by:

$$L_i = P_2^{-1} W_i \tag{35}$$

**Proof:** Let us consider the following quadratic Lyapunov function as follows:

$$V(e_1) = e_1^T P_2 e_1, \quad P_2 > 0 \tag{36}$$

Estimation error convergence is ensured if the following condition is guaranteed:

$$\dot{V}(e_1) = \dot{e}_1^T P_2 e_1 + e_1 P_2 \dot{e}_1 < 0 \tag{37}$$

By using (32), the condition (37) can be written as:

$$\dot{V}(e_1) = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi) e_1^T (\Gamma_{ij}^T P_2 + P_2 \Gamma_{ij}) e_1 < 0 \tag{38}$$

which is equivalent to the following stability conditions:

$$\begin{cases} \Gamma_{ii}^T P_2 + P_2 \Gamma_{ii} < 0 \qquad i = 1, \ldots, q \\ (\frac{\Gamma_{ij} + \Gamma_{ji}}{2})^T P_2 + P_2 (\frac{\Gamma_{ij} + \Gamma_{ji}}{2}) < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \emptyset \end{cases} \tag{39}$$

Letting $W_i = P_2 L_i$, it follows that (39) is equivalent to (34). From the Lypunov stability theory, if the LMI conditions (34) are satisfied, the estimated error equation (32) is exponentially asymptotically stable.

### 3.3. Dynamic Output Feedback Controller Design

In this section, we propose a novel approach for the separation principle theory of a class of T-S descriptor systems (1). A fuzzy dynamic output feedback controller for T-S descriptor system (1) is defined by:

$$u(\hat{x}) = u(\hat{X}_1, \hat{X}_2) = -\sum_{i=1}^{q} \mu_i(\xi)(K_{i1}\hat{X}_1 + K_{i2}\hat{X}_2) \tag{40}$$

in order that the closed-loop system be asymptotically stable where $K_{i1}$ and $K_{i2}$ are the control gains to be determined.

As in subsection 3.1, in order to be able to state the result in the form of LMIs, the key idea lies in the use of the equivalent control law of (40) given by:

$$u(\hat{X}_1, \hat{X}_2) = -\sum_{i=1}^{q} \mu_i(\xi) K_i \hat{X}_1 = \tilde{u}(\hat{X}_1) \tag{41}$$

where $K_i$ is defined in (17) which can be determined by Theorem 3 below and the control gains $K_{i1}$ and $K_{i2}$ used in (40) can be determined by equation (25).

Let

$$x_a = \begin{pmatrix} \hat{X}_1 \\ e_1 \end{pmatrix}, \; x_b = \begin{pmatrix} \hat{X}_2 \\ e_2 \end{pmatrix} \tag{42}$$

Combining (30), (32) and (41), we obtain the following augmented closed-loop system:

$$\begin{cases} \dot{x}_a = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)\Omega_{ij} x_a \\ x_b = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi)\Upsilon_{ij} x_a \end{cases} \tag{43}$$

Where

$$\begin{cases} \Omega_{ij} = \begin{pmatrix} \tilde{M}_{ij} & -L_i S_j \\ 0 & \Gamma_{ij} \end{pmatrix} \\ \Upsilon_{ij} = \begin{pmatrix} \tilde{Q}_{ij} & 0 \\ 0 & Q_i \end{pmatrix} \end{cases} \tag{44}$$

and $\tilde{M}_{ij}$, $\tilde{Q}_{ij}$ and $\Gamma_{ij}$ are defined in (20) and (33) respectively.

The following theorem provides the main result of this paper.

**Theorem 3.:** The closed-loop T-S model described by (43) is globally asymptotically stable if for a fixed scalar $\varepsilon > 0$, there exist matrices $Y_1 > 0$, $Y_2 > 0$, $U_i$, $W_i$, $i = 1, \ldots, q$ verifying the following LMIs:

$$\begin{cases} \begin{pmatrix} \Lambda_{ii}^{11} & 0 & I & 0 \\ 0 & \Lambda_{ii}^{22} & 0 & S_i^T W_i^T \\ I & 0 & -\varepsilon Y_2 & 0 \\ 0 & W_i S_i & 0 & -\varepsilon^{-1} Y_2 \end{pmatrix} < 0 \qquad i = 1,\dots,q \\ \begin{pmatrix} \Lambda_{ij}^{11} + \Lambda_{ji}^{11} & 0 & I & 0 \\ 0 & \Lambda_{ij}^{22} + \Lambda_{ji}^{22} & 0 & (W_i S_j + W_j S_i)^T \\ I & 0 & -\varepsilon Y_2 & 0 \\ 0 & W_i S_j + W_j S_i & 0 & -\varepsilon^{-1} Y_2 \end{pmatrix} < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \varnothing \end{cases} \tag{45}$$

where

$$\begin{cases} \Lambda_{ii}^{11} = Y_1^T M_i^T - U_i^T N_i^T + M_i Y_1 - N_i U_i \\ \Lambda_{ii}^{22} = M_i^T Y_2 - S_i^T W_i^T + Y_2 M_i - W_i S_i \\ \Lambda_{ij}^{11} = Y_1^T M_i^T - U_j^T N_i^T + M_i Y_1 - N_i U_j \\ \Lambda_{ji}^{11} = Y_1^T M_j^T - U_i^T N_j^T + M_j Y_1 - N_j U_i \\ \Lambda_{ij}^{22} = M_i^T Y_2 - S_j^T W_i^T + Y_2 M_i - W_i S_j \\ \Lambda_{ji}^{22} = M_j^T Y_2 - S_i^T W_j^T + Y_2 M_j - W_j S_i \end{cases} \tag{46}$$

The dynamic controller gains $K_i$ and $L_i$ are given by:

$$\begin{cases} K_i = U_i Y_1^{-1} \\ L_i = Y_2^{-1} W_i \end{cases} \tag{47}$$

**Proof:** To prove the convergence to zero of the state variable system (43), it suffices to prove that $x_a$ converges toward zero. Thus, let us consider the candidate Lyapunov function as follows:

$$V(x_a) = x_a^T P x_a \quad , \quad P > 0 \tag{48}$$

With

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tag{49}$$

The time derivative of the Lyapunov function (48) along the trajectories of the system (43) is obtained as:

$$\dot{V}(x_a) = \sum_{i,j=1}^{q} \mu_i(\xi)\mu_j(\xi) x_a^T (\Omega_{ij}^T P + P\Omega_{ij}) x_a \tag{50}$$

Therefore, we have the following stability conditions:

$$\begin{cases} \Omega_{ii}^T P + P\Omega_{ii} < 0 \qquad i = 1,\dots,q \\ (\frac{\Omega_{ij} + \Omega_{ji}}{2})^T P + P(\frac{\Omega_{ij} + \Omega_{ji}}{2}) < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \varnothing \end{cases} \tag{51}$$

which become from (44) and (49):

$$\begin{cases} \begin{pmatrix} \tilde{M}_{ii}^T P_1 + P_1 \tilde{M}_{ii} & -P_1 L_i S_i \\ -(L_i S_i)^T P_1 & \Gamma_{ii}^T P_2 + P_2 \Gamma_{ii} \end{pmatrix} < 0 \qquad i = 1,\dots,q \\ \begin{pmatrix} (\tilde{M}_{ij} + \tilde{M}_{ji})^T P_1 + P_1(\tilde{M}_{ij} + \tilde{M}_{ji}) & -P_1(L_i S_j + L_j S_i) \\ -(L_i S_j + L_j S_i)^T P_1 & (\Gamma_{ij} + \Gamma_{ji})^T P_2 + P_2(\Gamma_{ij} + \Gamma_{ji}) \end{pmatrix} < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \varnothing \end{cases} \tag{52}$$

Now multiplying the two inequalities given in (52) on the left and right by $\begin{pmatrix} (P_1^{-1})^T & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} P_1^{-1} & 0 \\ 0 & I \end{pmatrix}$ respectively and defining new variables $Y_1 = P_1^{-1}, Y_2 = P_2$, we obtain:

$$\begin{cases} \Lambda_1 = \begin{pmatrix} Y_1^T \tilde{M}_{ii}^T + \tilde{M}_{ii} Y_1 & -L_i S_i \\ -(L_i S_i)^T & \Gamma_{ii}^T Y_2 + Y_2 \Gamma_{ii} \end{pmatrix} < 0 \\ \Lambda_2 = \begin{pmatrix} Y_1^T (\tilde{M}_{ij} + \tilde{M}_{ji})^T + (\tilde{M}_{ij} + \tilde{M}_{ji}) Y_1 & -L_i S_j + L_j S_i) \\ -(L_i S_j + L_j S_i)^T & (\Gamma_{ij} + \Gamma_{ji})^T Y_2 + Y_2(\Gamma_{ij} + \Gamma_{ji}) \end{pmatrix} < 0 \\ i < j \; s.t. \; \mu_i \cap \mu_j \neq \varnothing \end{cases} \tag{53}$$

By considering the following:

$$\begin{cases} \Sigma_1 = \begin{pmatrix} Y_1^T \tilde{M}_{ii}^T + \tilde{M}_{ii} Y_1 & 0 \\ 0 & \Gamma_{ii}^T Y_2 + Y_2 \Gamma_{ii} \end{pmatrix} \\ \Sigma_2 = \begin{pmatrix} Y_1^T (\tilde{M}_{ij} + \tilde{M}_{ji})^T + (\tilde{M}_{ij} + \tilde{M}_{ji}) Y_1 & 0 \\ 0 & (\Gamma_{ij} + \Gamma_{ji})^T Y_2 + Y_2(\Gamma_{ij} + \Gamma_{ji}) \end{pmatrix} \\ Z_1 = \begin{pmatrix} 0 & L_i S_i \end{pmatrix} \\ Z_2 = \begin{pmatrix} 0 & L_i S_j + L_j S_i \end{pmatrix} \\ Z = \begin{pmatrix} -I & 0 \end{pmatrix} \end{cases} \tag{54}$$

Matrices $\Lambda_1$ and $\Lambda_2$ can be rewritten again as:

$$\begin{cases} \Lambda_1 = \Sigma_1 + Z_1^T Z + Z^T Z_1 \\ \Lambda_2 = \Sigma_2 + Z_2^T Z + Z^T Z_2 \end{cases} \tag{55}$$

**Lemma 1: Young's Inequality**

For any matrices $\tilde{X}$ and $\tilde{Z}$ with appropriate dimensions, the following property holds for any invertible matrix $J$ and scalar $\varepsilon > 0$:

$$\tilde{X}^T\tilde{Z}+\tilde{Z}^T\tilde{X} \le \varepsilon\tilde{X}^T J\tilde{X}+\varepsilon^{-1}\tilde{Z}^T J^{-1}\tilde{Z} \tag{56}$$

Applying Lemma 1 and taking $J = Y_2$, (55) becomes:

$$\begin{cases} \Lambda_1 \le \Sigma_1 - \begin{pmatrix} I & 0 \\ 0 & (L_iS_i)^T Y_2 \end{pmatrix}\begin{pmatrix} -\varepsilon Y_2 & 0 \\ 0 & -\varepsilon^{-1}Y_2 \end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ 0 & Y_2L_iS_i \end{pmatrix} \\ \Lambda_2 \le \Sigma_2 - \begin{pmatrix} I & 0 \\ 0 & (L_iS_j+L_jS_i)^T Y_2 \end{pmatrix}\begin{pmatrix} -\varepsilon Y_2 & 0 \\ 0 & -\varepsilon^{-1}Y_2 \end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ 0 & Y_2(L_iS_j+L_jS_i) \end{pmatrix} \end{cases} \tag{57}$$

Hence, using the Schur complement [21], the inequalities $\Lambda_1 < 0$ and $\Lambda_2 < 0$ hold if the following two matrix inequalities are satisfied.

$$\begin{cases} \begin{pmatrix} Y_1^T\tilde{M}_{ii}^T+\tilde{M}_{ii}Y_1 & 0 & I & 0 \\ 0 & \Gamma_{ii}^T Y_2+Y_2\Gamma_{ii} & 0 & (L_iS_i)^T Y_2 \\ I & 0 & -\varepsilon Y_2 & 0 \\ 0 & Y_2L_iS_i & 0 & -\varepsilon^{-1}Y_2 \end{pmatrix} < 0 \quad i=1,\dots,q \\ \begin{pmatrix} \Pi_{ij}^{11} & 0 & I & 0 \\ 0 & \Pi_{ij}^{22} & 0 & (L_iS_j+L_jS_i)^T Y_2 \\ I & 0 & -\varepsilon Y_2 & 0 \\ 0 & Y_2(L_iS_j+L_jS_i) & 0 & -\varepsilon^{-1}Y_2 \end{pmatrix} < 0 \quad i<j \ s\ t\ \mu_i\cap\mu_j\neq\varnothing \end{cases} \tag{58}$$

Where

$$\begin{cases} \Pi_{ij}^{11} = Y_1^T(\tilde{M}_{ij}+\tilde{M}_{ji})^T+(\tilde{M}_{ij}+\tilde{M}_{ji})Y_1 \\ \Pi_{ij}^{22} = (\Gamma_{ij}+\Gamma_{ji})^T Y_2+Y_2(\Gamma_{ij}+\Gamma_{ji}) \end{cases} \tag{59}$$

Then, from (20), (33) and the use of the changes of variables:

$$\begin{cases} U_i = K_iY_1 \\ W_i = Y_2L_i \end{cases} \tag{60}$$

we establish the LMI conditions (45) given in Theorem 3.

# 4. Application to an Inverted Pendulum System

To illustrate the effectiveness of the proposed method, we consider a nonlinear inverted pendulum system controlled by a separately excited direct-current (DC) motor [22]. Using the procedure of fuzzy model construction given in [23], the following T-S descriptor model with measurable premise variables can be derived:

$$\begin{cases} E\dot{x} = \sum_{i=1}^{2}\mu_i(\xi)(A_ix+Bu) \\ y = Cx \end{cases} \tag{61}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ is the state vector, $u$ denotes the control input voltage, $y = x_1$ is the measured output. $x_1 \in [-\pi\ \pi]$ is the angle of the pendulum, $x_2$ is the angular velocity, $x_3$ is the current of the motor.

$$\begin{cases} E=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_1=\begin{pmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -1 & -0.6 \end{pmatrix}, \\ A_2=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -0.6 \end{pmatrix},\ C=\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ and \\ \mu_1(\xi) = \dfrac{sin(x_1)}{x_1} \\ \mu_2(\xi) = 1-\dfrac{sin(x_1)}{x_1} \end{cases} \tag{62}$$

Note that, the application of the proposed dynamic controller (40) for inverted pendulum system requires that the above model (61) takes the form (12). To do so, let:

$X_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $X_2 = x_3$,

$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, with $rank(E) = 2$;

$$A_1 = \begin{pmatrix} A_{111} & A_{121} \\ A_{211} & A_{221} \end{pmatrix}, \ A_2 = \begin{pmatrix} A_{112} & A_{122} \\ A_{212} & A_{222} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}; \ C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

With:

$$A_{111} = \begin{pmatrix} 0 & 1 \\ 9.80 & 0 \end{pmatrix}, \ A_{112} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A_{121} = A_{122} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ A_{211} = A_{212} = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

$$A_{221} = A_{222} = -0.60, \ B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$B_2 = 1, \ C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ C_2 = 0$$

This show that the model (61) is a particular case of the system (1). Then, the fuzzy descriptor system (61) can be rewritten in the following equivalent form:

$$\begin{cases} \dot{X}_1 = \sum_{i=1}^{2} \mu_i(\xi)(M_i X_1 + N_i u) \\ X_2 = \sum_{i=1}^{2} \mu_i(\xi)(Q_i X_1 + R_i u) \\ y = \sum_{i=1}^{2} \mu_i(\xi)(S_i X_1 + G_i u) \end{cases} \quad (63)$$

With (see equation (8)):

$$M_1 = \begin{pmatrix} 0 & 1 \\ 9.80 & -1.6667 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1.6667 \end{pmatrix},$$

$$N_1 = N_2 = \begin{pmatrix} 0 \\ 1.6667 \end{pmatrix}, \quad Q_1 = Q_2 = \begin{pmatrix} 0 & -1.6667 \end{pmatrix},$$

$$S_1 = S_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ R_1 = R_2 = 1.6667, \ G_1 = G_2 = 0.$$

Consequently, using Theorem 3 with $\varepsilon = 0.01$, the following dynamic controller gains $K_1$, $K_2$, $L_1$ and $L_2$ were obtained:

$$K_1 = (14.6882 \quad 5.9640), \ K_2 = (9.8950 \quad 4.2533) \quad (64)$$

$$L_1 = \begin{pmatrix} 4.7155 \\ 7.6851 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 2.9333 \\ 3.3522 \end{pmatrix} \quad (65)$$

Simulation results with initial conditions

$$x_0 = [\frac{\pi}{10} \ 0 - 8.1456]^T, \ \hat{x}_0 = [\frac{\pi}{10} \ 0.05 - 8.2286]^T$$

are presented in Figures 1, 2 and 3. These simulation results show the performance of the proposed observer-based controller designed with the parameters $K_1$, $K_2$, $L_1$, $L_2$ given by (64) - (65). They show that the global asymptotic stability of the closed loop T-S system with the control law is satisfied. Moreover, Figures 2 and 3 show that the observer gives a good estimation of angular velocity and current of the motor.

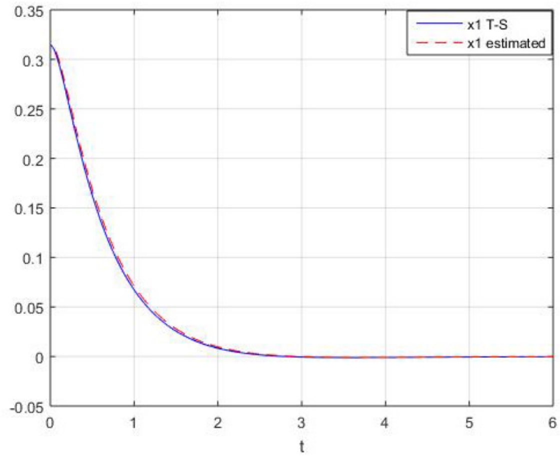


**Figure 1.** $x_1$ and $\hat{x}_1$ with fuzzy observer-based controller

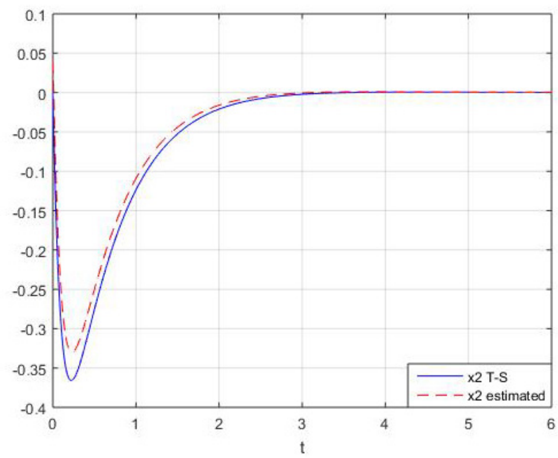


**Figure 2.** $x_2$ and $\hat{x}_2$ with fuzzy observer-based controller

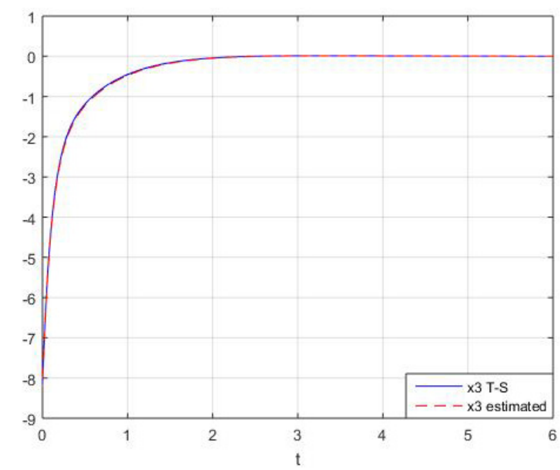


**Figure 3.** $x_3$ and $\hat{x}_3$. with fuzzy observer-based controller

On the other hand, using (25), we obtain the following control gains:

$$K_{11} = \begin{pmatrix} 0.4066 & -0.8072 \end{pmatrix},\ K_{12} = -0.5834,$$
$$K_{21} = \begin{pmatrix} 0.4413 & -0.7657 \end{pmatrix},\ K_{22} = -0.5732$$

Notice that with these numerical values of $K_{11}$, $K_{12}$, $K_{21}$ and $K_{22}$, we obtain exactly the same simulation results given in Figures 1, 2 and 3.

## 5. Conclusions

An observer-based fuzzy controller design approach for a class of T-S descriptor systems with measurable premise variables is proposed in this paper. The approach is based on the separation between dynamic and static relations in the descriptor model. The convergence of the closed-loop system is studied by using the Lyapunov theory and the stability conditions are given in terms of LMIs. Simulation results are given and demonstrate the good performance of the proposed dynamic controller.

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