

# On a Certain Subclass of Harmonic Multivalent Functions

Waggas Galib Atshan<sup>1,\*</sup>, Enaam Hadi Abd<sup>2,3</sup>

<sup>1</sup>Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya, Iraq

<sup>2</sup>Department of Computer, College of Science, University of Kerbala, Kerbala, Iraq

<sup>3</sup>Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

**Abstract** In this paper, we introduce a certain subclass of harmonic multivalent functions in the unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . We obtain some interesting properties, like, coefficient conditions, convex set, distortion theorems, weighted mean.

**Keywords** Harmonic multivalent functions, Distortion theorem, Convex set

## 1. Introduction

A continuous function  $f = u + iv$  is a complex-valued harmonic function in a domain  $D \subset \mathbb{C}$  if both  $u$  and  $v$  are real harmonic in  $D$ . In any simply connected domain  $D$ , we can write  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic in  $D$ . We call  $h$  the analytic part and  $g$  the co-analytic part of  $f$ . The harmonic function  $f = h + \bar{g}$  is sense-preserving and locally one-to-one in  $D$  if  $|h'(z)| > |g'(z)|$  in  $D$ . See Clunie and Sheil-Small [3].

For  $p \geq 1, k \in \mathbb{N}$ , denote by  $G(k, p)$  the class of functions  $f = h + \bar{g}$  that are harmonic multivalent and sense-preserving in the unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ , where  $h$  and  $g$  are defined by

$$\begin{aligned} h(z) &= z^p + \sum_{k=p+1}^{\infty} a_k z^k, \\ g(z) &= \sum_{k=p}^{\infty} b_k z^k, \quad |b_1| < 1 \end{aligned} \quad (1)$$

which are analytic and multivalent functions in  $U$ .

Also, denote by  $\bar{G}(k, p)$  subclass of  $G(k, p)$  consisting of harmonic functions  $f = h + \bar{g}$  where  $h$  and  $g$  are of the form

$$\begin{aligned} h(z) &= z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, \\ g(z) &= -\sum_{k=p}^{\infty} |b_k| z^k, \quad |b_1| < 1 \end{aligned} \quad (2)$$

In 1984, Clunie and Sheil-Small [3] investigated the class  $S_H$  and as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on  $S_H$  and its subclasses. For more basic results one may refer to the following standard

introductory text book by Duren [4], see also Ahuja [1], Jahangiri et al. [5] and Ponnusamy and Rasila ([6], [7]).

The convolution of two functions of form

$$\begin{aligned} f(z) &= z^p + \sum_{k=p+1}^{\infty} a_k z^k \text{ and} \\ F(z) &= z^p + \sum_{k=p+1}^{\infty} A_k z^k \text{ is defined as} \\ (f * F)(z) &= f(z) * F(z) = z^p + \sum_{k=p+1}^{\infty} a_k A_k z^k \end{aligned} \quad (3)$$

Let  $Gp(k, p, \lambda, \beta, \mu)$  denote the subclass of  $G(k, p)$  consisting of functions  $f = h + \bar{g} \in G(k, p)$  that satisfy the condition

$$\begin{aligned} \operatorname{Re} \left\{ (1 - \lambda) + \frac{\beta \lambda z^p + (1 - \lambda) z^2 (f(z))''}{z(f(z))'} \right\} &> (\beta \lambda + (1 - \lambda) p^2) \mu, \\ 0 \leq \lambda \leq 1, \mu &< \frac{1}{p}, \beta \geq 0, p \geq 2 \end{aligned} \quad (4)$$

Define  $\bar{G}p(k, p, \lambda, \beta, \mu) = Gp(k, p, \lambda, \beta, \mu) \cap \bar{G}(k, p)$ .

**Lemma (1) [2]:** Let  $\alpha \geq 0$ . Then  $\operatorname{Re}\{w\} > \alpha$  if and only if  $|w - (1 + \alpha)| < |w + (1 - \alpha)|$ , where  $w$  be any complex number.

## 2. Coefficient Estimates

We begin with a sufficient condition for the function in  $f$  to be the class  $Gp(k, p, \lambda, \beta, \mu)$ .

**Theorem (2.1):** Let  $f = h + \bar{g}$  defined in (1). If

$$\begin{aligned} \sum_{k=p+1}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k] |a_k| \\ + \sum_{k=p}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k] |b_k| - p \leq 0, \end{aligned} \quad (5)$$

Where  $0 < \lambda \leq 1, \mu < \frac{1}{p}, p \geq 2$ , then  $f$  is harmonic Multivalent sense-preserving in  $U$  and  $f \in Gp(k, p, \lambda, \beta, \mu)$ .

**Proof:** Let

$$w(z) = \left\{ (1 - \lambda) + \frac{\beta \lambda z^p + (1 - \lambda) z^2 (f(z))''}{z(f(z))'} \right\} = \frac{A(z)}{B(z)}$$

By lemma (1), we must show that (4) holds true. It suffices to show that

\* Corresponding author:

waggashnd@gmail.com (Waggas Galib Atshan)

Published online at <http://journal.sapub.org/ajcam>

Copyright © 2016 Scientific & Academic Publishing. All Rights Reserved

$$|A(z) - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)B(z)| - |A(z) + (1 - (\beta\lambda + (1 - \lambda)p^2)\mu)B(z)| \leq 0$$

Substituting for  $w$  and resorting to simple calculation, we find

$$\begin{aligned}
& |A(z) - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)B(z)| \\
&= |(1 - \lambda)z(f(z))' + \beta\lambda z^p + (1 - \lambda)[z^2(f(z))''] - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)[z(f(z))']| \\
&= \left| (1 - \lambda)z[(h(z))' + \overline{(g(z))'}] + \beta\lambda z^p + (1 - \lambda)z^2[(h(z))'' + \overline{(g(z))'']}] - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)z[(h(z))' + \overline{(g(z))'}] \right| \\
&= \left| (1 - \lambda)pz^p + (1 - \lambda) \sum_{k=p+1}^{\infty} k a_k z^k \right. \\
&\quad + (1 - \lambda) \sum_{k=p}^{\infty} k b_k z^k + \beta\lambda z^p + (1 - \lambda)p(p - 1)z^p \\
&\quad + (1 - \lambda) \sum_{k=p+1}^{\infty} k(k - 1)a_k z^k \\
&\quad + (1 - \lambda) \sum_{k=p}^{\infty} k(k - 1)b_k z^k - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)pz^p \\
&\quad \left. - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu) \sum_{k=p+1}^{\infty} k a_k z^k - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu) \sum_{k=p}^{\infty} k b_k z^k \right| \\
&= |[ (1 - \lambda)p + (1 - \lambda)p(p - 1) + \beta\lambda - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)p ] z^p \\
&\quad + \sum_{k=p+1}^{\infty} [(1 - \lambda)k + (1 - \lambda)k(k - 1) - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)k] a_k z^k \\
&\quad + \sum_{k=p}^{\infty} [(1 - \lambda)k + (1 - \lambda)k(k - 1) - (1 + (\beta\lambda + (1 - \lambda)p^2)\mu)k] b_k z^k | \\
&= |[(\beta\lambda + (1 - \lambda)p^2)(1 - p\mu) - p] z^p + \sum_{k=p+1}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k - k] a_k z^k \\
&\quad + \sum_{k=p}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k - k] b_k z^k | \\
&\leq [(\beta\lambda + (1 - \lambda)p^2)(1 - p\mu) - p] |z|^p + \sum_{k=p+1}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k - k] |a_k| |z|^k \\
&\quad + \sum_{k=p}^{\infty} [(1 - \lambda)k(k - p^2\mu) - \beta\lambda\mu k - k] |b_k| |z|^k \\
&|A(z) + (1 - (\beta\lambda + (1 - \lambda)p^2)\mu)B(z)| \\
&= |(1 - \lambda)z(f(z))' + \beta\lambda z^p + (1 - \lambda)[z^2(f(z))''] + (1 - (\beta\lambda + (1 - \lambda)p^2)\mu)[z(f(z))']| \\
&= \left| (1 - \lambda)z[(h(z))' + \overline{(g(z))'}] + \beta\lambda z^p + (1 - \lambda)z^2[(h(z))'' + \overline{(g(z))'']}] \right. \\
&\quad \left. + (1 - (\beta\lambda + (1 - \lambda)p^2)\mu)z[(h(z))' + \overline{(g(z))'}] \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| (1-\lambda)pz^p + (1-\lambda) \sum_{k=p+1}^{\infty} ka_kz^k \right. \\
&\quad + (1-\lambda) \sum_{k=p}^{\infty} kb_kz^k + \beta\lambda z^p + (1-\lambda)p(p-1)z^p \\
&\quad + (1-\lambda) \sum_{k=p+1}^{\infty} k(k-1)a_kz^k \\
&\quad + (1-\lambda) \sum_{k=p}^{\infty} k(k-1)b_kz^k + (1-(\beta\lambda + (1-\lambda)p^2)\mu)pz^p \\
&\quad \left. + (1-(\beta\lambda + (1-\lambda)p^2)\mu) \sum_{k=p+1}^{\infty} ka_kz^k + (1-(\beta\lambda + (1-\lambda)p^2)\mu) \sum_{k=p}^{\infty} kb_kz^k \right| \\
&= |[(1-\lambda)p + (1-\lambda)p(p-1) + \beta\lambda + (1-(\beta\lambda + (1-\lambda)p^2)\mu)p]z^p \\
&\quad + \sum_{k=p+1}^{\infty} [(1-\lambda)k + (1-\lambda)k(k-1) + (1-(\beta\lambda + (1-\lambda)p^2)\mu)k]a_kz^k \\
&\quad + \sum_{k=p}^{\infty} [(1-\lambda)k + (1-\lambda)k(k-1) + (1-(\beta\lambda + (1-\lambda)p^2)\mu)k]b_kz^k| \\
&= |[(\beta\lambda + (1-\lambda)p^2)(1-p\mu) + p]z^p + \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]a_kz^k \\
&\quad + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]b_kz^k| \\
&\geq [(\beta\lambda + (1-\lambda)p^2)(1-p\mu) + p]|z|^p - \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|a_k||z|^k \\
&\quad - \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|b_k||z|^k \\
&= |A(z) - (1 + (\beta\lambda + (1-\lambda)p^2)\mu)B(z)| = |A(z) + (1 - (\beta\lambda + (1-\lambda)p^2)\mu)B(z)| \\
&\leq [(\beta\lambda + (1-\lambda)p^2)(1-p\mu) - p] + \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|a_k| \\
&\quad + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|b_k| \\
&= - \left[ [(\beta\lambda + (1-\lambda)p^2)(1-p\mu) + p] - \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|a_k| \right. \\
&\quad \left. - \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|b_k| \right] \\
&= -2p + 2 \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + 2 \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k| \\
&= \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k| - p \leq 0
\end{aligned}$$

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{X_k}{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]} z^k + \sum_{k=p}^{\infty} \frac{\bar{Y}_k}{\lambda[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]} (\bar{z})^k,$$

where

$$\sum_{k=p+1}^{\infty} |X_k| + \sum_{k=p}^{\infty} |\bar{Y}_k| = p$$

$$\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_k| = \sum_{k=p+1}^{\infty} |X_k| + \sum_{k=p}^{\infty} |\bar{Y}_k| = p$$

**Theorem (2.2):** Let  $f = h + \bar{g}$  with  $h$  and  $g$  given by (2). Then  $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$  if and only if

$$\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_k| \leq p. \quad (6)$$

**Proof:** From theorem (2.1) to prove the necessary part, let us assume that  $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$  using (4), we get

$$\begin{aligned} & \operatorname{Re} \left\{ (1-\lambda) + \frac{\beta\lambda z^p + (1-\lambda)z^2(f(z))''}{z(f(z))'} \right\} \\ &= \operatorname{Re} \left\{ \frac{(1-\lambda)z(f(z))' + \beta\lambda z^p + (1-\lambda)z^2(f(z))''}{z(f(z))'} \right\} \\ &= \operatorname{Re} \left\{ \frac{(1-\lambda)z \left[ (h(z))' + \overline{(g(z))'} \right] + \beta\lambda z^p + (1-\lambda)z^2 \left[ (h(z))'' + \overline{(g(z))''} \right]}{z \left[ (h(z))' + \overline{(g(z))'} \right]} \right\} \\ &= \operatorname{Re} \left\{ \frac{[(1-\lambda)p^2 + \beta\lambda]z^p - \sum_{k=p+1}^{\infty} [(1-\lambda)k^2] |a_k| z^k - \sum_{k=p}^{\infty} [(1-\lambda)k^2] |b_k| (\bar{z})^k}{pz^p - \sum_{k=p+1}^{\infty} k |a_k| z^k - \sum_{k=p}^{\infty} k |b_k| (\bar{z})^k} \right\} \\ &> (\beta\lambda + (1-\lambda)p^2)\mu. \end{aligned}$$

If we choose  $z$  to be real and let  $z \rightarrow 1^-$ , we obtain the condition (6) and the proof is complete.

### 3. Convex Set

**Theorem (3.1):** The class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$  is convex set.

**Proof:** Let the function  $f_i(z) (i = 1, 2)$  be in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$ . It is sufficient to show that the function  $H$  defined by:

$$H(z) = (1-e)f_1(z) + ef_2(z), \quad (0 \leq e \leq 1) \quad (7)$$

is in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$  we have

$$f_i = z^p - \sum_{k=p+1}^{\infty} |a_{k,i}| z^k - \sum_{k=p}^{\infty} |b_{k,i}| \bar{z}^k.$$

Since for  $e$  ( $0 \leq e \leq 1$ )

$$H(z) = z^p - \sum_{k=p+1}^{\infty} ((1-e)|a_{k,1}| - e|a_{k,2}|) z^k - \sum_{k=p}^{\infty} ((1-e)|b_{k,1}| - e|b_{k,2}|) \bar{z}^k.$$

In view of Theorem (2.1), we have

$$\begin{aligned} & \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] ((1-e)|a_{k,1}| - e|a_{k,2}|) + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] ((1-e)|b_{k,1}| - e|b_{k,2}|) \\ &= (1-e) \left[ \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_{k,1}| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_{k,1}| \right] \end{aligned}$$

$$+ e \left[ \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_{k,2}| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_{k,2}| \right]$$

$$\leq (1-e)p + ep = p.$$

Hence,  $(z) \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ . This completes the proof.

## 4. Distortion and Covering

The next Theorem is on the distortion and covering bounds for functions in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$ .

**Theorem (4.1):** If  $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ ,  $|z| = r < 1$ , then

$$|f(z)| \geq (1 - |b_p|)r^p - \frac{p[1-|b_p|]}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} r^{p+1}, \quad (8)$$

and

$$|f(z)| \leq (1 + |b_p|)r^p + \frac{p[1-|b_p|]}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} r^{p+1}. \quad (9)$$

**Proof:** Assume that  $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ . Then by Theorem (2.2), we obtain

$$\begin{aligned} |f(z)| &= \left| z^p - \sum_{k=p+1}^{\infty} |a_k| z^k - \sum_{k=p}^{\infty} |b_k| (\bar{z})^k \right| \geq (1 - |b_p|)r^p - \sum_{k=p+1}^{\infty} (|a_k| + |b_k|)r^k \\ &\geq (1 - |b_p|)r^p - \sum_{k=p+1}^{\infty} (|a_k| + |b_k|)r^{p+1} \\ &= (1 - |b_p|)r^p - \frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} \sum_{k=p+1}^{\infty} (p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu] (|a_k| + |b_k|)r^{p+1} \\ &\geq (1 - |b_p|)r^p - \frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} \sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] (|a_k| + |b_k|)r^{p+1} \\ &\geq (1 - |b_p|)r^p - \frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} [p - p|b_p|]r^{p+1} \\ &= (1 - |b_p|)r^p - \frac{p[1-|b_p|]}{(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]} r^{p+1}. \end{aligned}$$

Relation (9) can be proved by using the similar statements.

The covering result given in corollary (4.2) follows from the inequality (9) of this Theorem.

**Corollary(4.2):** If  $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ , then

$$\left\{ w : |w| < (1 - |b_p|)r^p - \frac{p[1-|b_p|]}{[(p+1)[(1-\lambda)(1+p-p^2\mu) - \beta\lambda\mu]]} \right\} \subset f(U).$$

## 5. Weighted Mean

**Definition (5.1):** Let  $f$  and  $F \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ , where

$$f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k - \sum_{k=p}^{\infty} |b_k| z^k \quad \text{and} \quad F(z) = z^p - \sum_{k=p+1}^{\infty} |A_k| z^k - \sum_{k=p}^{\infty} |B_k| z^k.$$

Then the weighted mean  $E_i(z)$  of  $f$  and  $g$  is given by

$$E_i(z) = \frac{1}{2} [(1-i)f(z) + (1+i)F(z)], \quad 0 < i < 1.$$

In the theorem below, we will show the weighted mean for this class:

**Theorem (5.2):** If  $f$  and  $F$  be in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$ , then the weighted mean of  $f$  and  $F$  is also in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$ .

**Proof:** By Definition (5.1), we have

$$\begin{aligned} E_i(z) &= \frac{1}{2}[(1-i)f(z) + (1+i)g(z)] \\ E_i(z) &= \frac{1}{2} \left[ (1-i) \left( z^p - \sum_{k=p+1}^{\infty} |a_k| z^k - \sum_{k=p}^{\infty} |b_k| z^k \right) + (1+i) \left( z^p - \sum_{k=p+1}^{\infty} |A_k| z^k - \sum_{k=p}^{\infty} |B_k| \bar{z}^k \right) \right] \\ &= \frac{1}{2} \left[ \left( z^p - \sum_{k=p+1}^{\infty} (1-i)|a_k| z^k - \sum_{k=p}^{\infty} (1-i)|b_k| z^k \right) + \left( z^p - \sum_{k=p+1}^{\infty} (1+i)|A_k| z^k - \sum_{k=p}^{\infty} (1+i)|B_k| \bar{z}^k \right) \right] \\ &= z^p - \sum_{k=p+1}^{\infty} \frac{1}{2} [(1-i)|a_k| + (1+i)|A_k|] z^k - \sum_{k=p}^{\infty} \frac{1}{2} [(1-i)|b_k| + (1+i)|B_k|] \bar{z}^k. \end{aligned}$$

Since  $f$  and  $F$  are in the class  $\overline{Gp}(k, p, \lambda, \beta, \mu)$  so by Theorem (2.2), we get

$$\frac{\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_k|}{p} \leq 1$$

and

$$\frac{\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |A_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |B_k|}{p} \leq 1$$

Then

$$\begin{aligned} &= \sum_{k=p+1}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p} \left( \frac{1}{2} [(1-i)|a_k| + (1+i)|A_k|] \right) \\ &\quad + \sum_{k=p}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p} \left( \frac{1}{2} [(1-i)|b_k| + (1+i)|B_k|] \right) \\ &= \frac{1}{2} (1-i) \left[ \frac{\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |b_k|}{p} \right] \\ &\quad + \frac{1}{2} (1+i) \left[ \frac{\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |A_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k] |B_k|}{p} \right] \\ &\leq \frac{1}{2} (1-i) + \frac{1}{2} (1+i) = 1. \end{aligned}$$

## REFERENCES

- 
- [1] O. P. Ahuja, Palnar harmonic univalent and related mapping, J. Inequal. Pure Appl. Math., 6, No. 4 (2005), 122, 1-18
  - [2] E. S. Aqlan, Some Problems Connected with Geometric Function Theory, Ph.D. Thesis (2004), Pune University, Pune.
  - [3] J. Clunie, T. Shell-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A1. Math., 9, No. 3 (1984), 3-25.
  - [4] P. Duren, Harmonic Mappings in the Plane, Cambridge Tract in Mathematics, 156, Cambridge University Press, Cambridge (2004).
  - [5] J. M. Jahangiri, Chan Kim Young, H. M. Srivastava, Construction of a certain class of harmonic close to convex functions associated with the alexander integral transform, Integral Transform Spec. Funct., 14, No. 3 (2003), 237-242.
  - [6] S. Ponnusamy, A. Rasila, Planar harmonic mapping, RMS Mathematics Newsletter, 17, No. 2 (2007), 40-57.
  - [7] S. Ponnusamy, A. Rasila, Planar harmonic and quasiconformal mappings, RMS Mathematics Newsletter, 17, No. 3 (2007), 85-101.