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Nth Order Pulse Solitary Wave Solution and Modulational Instability in the Boussinesq Equation

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Abstract We look for the nth order exact pulse solitary wave solutions of the Boussinesq equation. For this reason, we define and find out all the solutions of the generalized Boussinesq equation. We later on deduce the solutions of classical Boussinesq equation. The modulational instability of the pulse solitary wave solutions obtained is also investigated in detail by using the Bogning- Djeumen Tchaho- Kofané method.

Keywords Generalized Boussinesq equation, Solitary waves, Modulational Instability

1. Introduction

The French physicist Joseph Boussinesq appears among many precursors of the nonlinear physics to the 19th century. His talent allowed him to do many works in the domain of the soil mechanics and especially of the fluid mechanics. He succeeded in modeled a nonlinear partial differential equation that describes the dynamics of propagation of the waves in little deep waters. This equation that carries his name is an equation that is going to revolutionize the domain of the hydrodynamics [1-7]. So, Apart from Korteweg – de Vries (KdV) equation, one of the equations that better describes the dynamics of shallow water waves in the oceanography context is the Boussinesq equation. If the KdV equation is definitely well known through its solitary wave solution, the Boussinesq equation continues to reveal the supplementary information according to approximations, it can be suggested. As all other nonlinear partial differential equation, there exist various methods of analysis the Boussinesq equation. Some of the commonly used techniques are the variational iteration method, the semi- inverse variational principle, the G/G- expansion method, the exp-function method, the F-expansion methods and many others [8-15]. Beyond several methods that can contribute to solving the nonlinear partial differential equations of this order, many authors made different analyses on the Boussinesq equations [16-18]. Our work in this article is to determine the nth order solitary wave solutions of the form $U_n(x,t) = a\sec h^n(\alpha x - \alpha_0 t)$ of the Boussinesq equation with cubic non linearity and establish their modulational instability criteria with the help of Bogning – Djeumen Tchaho – Kofané method (BDkm) [18-23]. Thus, this work is organized as follow: In section two, we define the generalized Boussinesq equation and obtain the range equations of coefficients. Section three analyses and proposes different solutions of the generalized and classical Boussinesq equation. In section four we study the modulational instability of the obtained solitary wave solution in the generalized Boussinesq equation and in the classical Boussinesq equation. Finally, we end the work by a general conclusion.

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2. Method and Range Equations Deriving from the Boussinesq Equation

Boussinesq equation in its classical form is given by

$$U_{tt} - U_{xx} - U_{xxxx} - 3\left(U^3\right)_{xx} = 0, \qquad (1)$$

where the parameters subscript t and the subscript x stand respectively for the derivative with respect to time and space. But in our analysis, we are going to consider Boussinesq equation under its generalized form

$$n_0 U_{tt} + n_1 U_{xx} + n_2 U_{xxxx} + n_3\left(U^3\right)_{xx} = 0, \qquad (2)$$

where the coefficients n_0, n_1, n_2 and n_3 are the real numbers to be determined. Thus, looking for the solution of equation (2) in the form

$$U_n(x,t) = a\sec h^n(\alpha x - \alpha_0 t), \qquad (3)$$

where a, n, α and α_0 are real to be determined, we obtain via the transformations of reference [19, 20, 24], the following equation

$$\begin{aligned}&\left[a\alpha_0^2n^2n_0+a\alpha^2n^2n_1+a\alpha^4n^4n_2\right]\sec h^n\xi\\&-\left[a\alpha_0^2\left(n^2+n\right)n_0+a\alpha^2\left(n^2+n\right)n_1+a\alpha^4\left(2n^4+6n^3+8n^2+4n\right)n_2\right]\sec h^{n+2}\xi\\&+a\alpha^4\left(n^4+6n^3+11n^2+6n\right)n_2\sec h^{n+4}\xi\\&+9a^3\alpha^2n^2n_3\sec h^{3n}\xi-a^3\alpha^2\left(9n^2+3n\right)n_3\sec h^{3n+2}\xi=0,\end{aligned} \tag{4}$$

where $\xi=\alpha x-\alpha_0t$. Equation (4) is called general range equations of coefficients for any *n*. This equation is the main equation which will determined all the possibilities to obtain the solutions of equation (2).

3. Analysis of Equation (4) and Results

Equation (4) obtained to the following analysis: we check the values of n for which the terms in $\sinh\xi\sec h^i\xi$ with $i=n$, $n+2$, $n+4$, $3n$, and $3n+2$ are identical. The terms in $\sec h^n\xi$ and $\sec h^{3n}\xi$ are identical for $n=0$, the terms in $\sec h^{n+2}\xi$ and $\sec h^{3n+2}\xi$ for $n=0$; the terms in $\sec h^n\xi$ and $\sec h^{3n+2}\xi$ are identical for $n=-1$; the terms in $\sec h^{n+2}\xi$ and $\sec h^{3n}\xi$ are identical for $n=1$, the terms in $\sec h^{n+4}\xi$ and $\sec h^{3n+2}\xi$ are identical for $n=1$; the terms in $\sec h^{n+4}\xi$ and $\sec h^{3n}\xi$ are identical for $n=2$. These values are obtained just by equalizing different values of i (for example $n=3n$, $n=3n+2$, $n+2=3n$, $n+2=3n+2$, $n+4=3n$ and $n+4=3n+2$). So for $n=0$, $n=-1$ and $n=2$, equation(4) admits only the trivial solutions. The acceptable solutions of equation (4) are obtained only for $n=1$, equation (4) becomes.

$$\begin{aligned}&\left(a\alpha_0^2n_0+a\alpha^2n_1+a\alpha^4n_2\right)\sec h\xi\\&+\left(9a^3\alpha^2n_3-2a\alpha_0^2n_0-2a\alpha^2n_1-20a\alpha^4n_2\right)\sec h^3\xi\\&+\left(24a\alpha^4n_2-12a^3\alpha^2n_3\right)\sec h^5\xi=0.\end{aligned} \tag{5}$$

By identifying the different term of equation (5) to zero, we obtain the following set of equations

$$a\alpha_0^2n_0+a\alpha^2n_1+a\alpha^4n_2=0, \tag{6}$$

$$9a^3\alpha^2n_3-2a\alpha_0^2n_0-2a\alpha^2n_1-20a\alpha^4n_2=0, \tag{7}$$

and

$$24a\alpha^4n_2-12a^3\alpha^2n_3=0. \tag{8}$$

From equation (8), we obtain

$$a^2=2\alpha^2\frac{n_2}{n_3};\ \text{with}\ \ n_2n_3\succ 0. \tag{9}$$

Inserting equation (9) into equation (7) leads to equation (6), then the problem reduces to the resolution of equations (6) and (9). So considering equation (6), we see that it is a polynomial equation of order four in α. Taking into account the fact that the equation (2) must be of the Boussinesq form such that $n_0\neq 0$, $n_1\neq 0$, $n_2\neq 0$ and $n_3\neq 0$ with $a\neq 0$, the discriminant is $\Delta=n_1^2-4n_2n_0$; According to the values of Δ, we look for the different solutions of equation (6).

- For $\Delta=n_1^2-4n_2n_0\succ 0$, we obtain

$$\alpha^2 = \frac{-n_1 \pm \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{2n_2}. \tag{10}$$

The relation (10) is defined if its right hand side is positive. So, for $n_1 \prec 0$, $n_2 \prec 0$ and $n_0 \succ 0$

$$\alpha = \left(\frac{-n_1 - \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{2n_2} \right)^{\frac{1}{2}}, \tag{11}$$

Such that from equation (9) we deduce the expression of a as

$$a = \left(\frac{-n_1 - \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{n_3} \right)^{\frac{1}{2}}; \text{ with } n_3 \succ 0 \tag{12}$$

Substituting α and a by their respective expression given by equations (11) and (12) in the ansatz (3), we obtain

$$U(x,t) = \left(\frac{-n_1 - \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{n_3} \right)^{\frac{1}{2}} \times \operatorname{sec}h \left[\left(\frac{-n_1 - \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{2n_2} \right)^{\frac{1}{2}} x - \alpha_0 t \right]. \tag{13}$$

For $n_1 \succ 0$, $n_2 \prec 0$ and $n_0 \prec 0$, from equation (10), we deduce the expression of α as

$$\alpha = \left(\frac{-n_1 + \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{2n_2} \right)^{\frac{1}{2}}. \tag{14}$$

Taking into account equation (14) in equation (9), we obtain

$$a = \left(\frac{-n_1 + \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{n_3} \right)^{\frac{1}{2}}; \text{ with } n_3 \prec 0. \tag{15}$$

With equation (14) and (15), the ansatz (3) reads

$$U(x,t) = \left(\frac{-n_1 + \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{n_3} \right)^{\frac{1}{2}} \times \operatorname{sec}h \left[\left(\frac{-n_1 + \sqrt{n_1^2 - 4n_0 n_2 \alpha_0^2}}{2n_2} \right)^{\frac{1}{2}} x - \alpha_0 t \right]. \tag{16}$$

In case of equation (13) like equation (16), α_0 must be chosen such that these solutions are possible.

- For $\Delta = n_1^2 - 4n_2 n_0 = 0$, we obtain

$$\alpha = \left(\frac{-n_1}{2n_2} \right)^{\frac{1}{2}}; \text{ with } n_1 n_2 \prec 0, \tag{17}$$

and

$$a = \left(\frac{-n_1}{n_3} \right)^{\frac{1}{2}}; \text{ with } n_1 n_3 \prec 0. \tag{18}$$

Then the solution of equation (2) in this case is given by

$$U(x,t)=\left(\frac{-n_1}{n_3}\right)^{\frac{1}{2}}\sec h\left[\left(\frac{-n_1}{2n_2}\right)^{\frac{1}{2}}x-\alpha_0 t\right]. \tag{19}$$

- For $\Delta = n_1^2 - 4n_2n_0 \prec 0$, we obtain from equation (6)

$$\alpha^2=\frac{-n_1\pm i\sqrt{4n_0n_2\alpha_0^2-n_1^2}}{2n_2}, \tag{20}$$

and

$$a^2=\frac{-n_1\pm i\sqrt{4n_0n_2\alpha_0^2-n_1^2}}{n_3}. \tag{21}$$

In this case, the parameters α and a are complexes and we can also deduce the series of solutions linked.

From the above analyses, we deduce the analytical solution of the classical Boussinesq equation (1) by setting $n_0=1$, $n_1=-1$, $n_2=-1$ and $n_3=-3$. Thus, from equation (13) we obtain

$$U(x,t)=\left(\frac{-1-\sqrt{1+4\alpha_0^2}}{3}\right)^{\frac{1}{2}}\sec h\left[\left(\frac{-1+\sqrt{1+4\alpha_0^2}}{2}\right)^{\frac{1}{2}}x-\alpha_0 t\right];\ \text{with } \alpha_0\in\mathbb{R}. \tag{22}$$

We also have from equation (19),

$$U(x,t)=\left(\frac{1}{3}\right)^{\frac{1}{2}}\sec h\left[\left(\frac{1}{2}\right)^{\frac{1}{2}}x-\alpha_0 t\right]. \tag{23}$$

4. Modulational Instability of Bright Solitary Wave in Boussinesq Equation

We analyse the modulational instability of the generalized Boussinesq equation (2), by looking for its solution in the form

$$U(x,t)=(a+\delta)\sec h\xi, \tag{24}$$

where δ is the smallest term of perturbation which varies as a function of time and space. The solution here is chosen in the form given by equation (24) because we have verified in the preceeding section that the generalized Boussinesq equation admits the $\sec h^n\xi$ -solution only in the case where $n=1$. Thus, inserting equation (24) in equation (2), we obtain

$$\begin{aligned}&F_1\sec h\xi+F_2\sec h^2\xi+F_3\sec h^3\xi+F_4\sec h^5\xi\\&+G_1\sinh\xi\sec h^2\xi+G_2\sinh\xi\sec h^3\xi\\&+G_3\sinh\xi\sec h^4\xi=0,\end{aligned} \tag{25}$$

with

$$\begin{aligned}F_1=&\,n_0\delta_{tt}+n_2\delta_{xxxx}+8n_2\alpha^2\delta_{xx}+\left(n_0\alpha^2+n_1\alpha^2+\alpha^4n_2\right)\delta\\&+n_0\alpha_0^2+n_1\alpha^2+n_2\alpha^4,\end{aligned} \tag{26}$$

$$F_2=-2n_2\alpha^2\delta_{xx}, \tag{27}$$

$$\begin{aligned}F_3 &= \left(3n_3a^2-6n_2\alpha^2\right)\delta_{xx}-6n_2\alpha^2\delta_x+6n_3a\delta_{xx}\delta\\&+6n_3a\delta_x^2+3n_3\delta_{xx}\delta^2+6n_3\delta_x^2\delta\\&+\left(-2n_0\alpha_0^2-2n_1\alpha^2-20n_2\alpha^4-9n_3\alpha^2a^2+3n_3a^2\right)\delta\\&+\left(-3n_3\alpha^2+3n_2a-9n_3\alpha^2a\right)\delta^2\\&+n_3\delta^3-2n_0\alpha_0^2a-20n_2\alpha^4a+9n_3\alpha^2a^3-2n_1\alpha^2a,\end{aligned} \tag{28}$$

$$\begin{aligned}F_4 &= \left(24n_2\alpha^4-36n_3a^2\alpha^2\right)\delta-36n_3\alpha^2a\delta^2-12n_3\alpha^2\delta^3\\&+12\alpha^2a\left(2n_2\alpha^2-n_3a^2\right),\end{aligned} \tag{29}$$

$$G_1 = 2n_0\alpha_0\delta_t-4n_2\alpha\delta_{xxx}-\left(2n_1\alpha+6n_2\alpha^3\right)\delta_x, \tag{30}$$

$$G_2 = 2n_2\alpha^2\delta_x, \tag{31}$$

and

$$G_3 = \left(24n_2\alpha^3-18n_3\alpha a^2\right)\delta_x-36n_3a\delta_x\delta-18\alpha n_3\delta_x\delta^2. \tag{32}$$

Equation (25) is verified if and only if $F_i = 0,$ for $i = 1, 2, 3, 4$ and $G_i = 0,$ for $i = 1, 2, 3$. According to the analysis made in references [19, 20, 24], the equation which is susceptible to express better the perturbation is given by the term in $\sec h\xi$, but we also take into account that δ is a very small parameter. So the term in $\sec h\xi$ leads to

$$n_0\delta_{tt}+n_2\delta_{xxxx}+8n_2\alpha^2\delta_{xx}+\left(n_0\alpha^2+n_1\alpha^2+\alpha^4n_2\right)\delta+n_0\alpha_0^2+n_1\alpha^2+n_2\alpha^4 = 0. \tag{33}$$

Taking into account equation (6) in equation (33), it becomes

$$n_0\delta_{tt}+n_2\delta_{xxxx}+8n_2\alpha^2\delta_{xx}+\left(n_0\alpha^2+n_1\alpha^2+\alpha^4n_2\right)\delta = 0. \tag{34}$$

Equation (34) is the main equation of perturbation that will enables us to study effectively the modulational instability of solitary wave solution $U(x,t) = a\sec h\xi$. If we suppose that the term of perturbation is given by

$$\delta = A\exp-i\left(kx-\omega t\right), \tag{35}$$

where A is an arbitrary constant, equation (34) leads to

$$\omega = \pm\sqrt{\frac{n_2}{n_0}\left[\left(k^2-4\alpha^2\right)^2+\left(\frac{n_0+n_1}{n_2}\right)\alpha^2-15\alpha^4\right]}. \tag{36}$$

- If $n_0n_2 \succ 0$ and $(n_0+n_1)n_2 \succ 0$, the dispersion relation (36) is valid for $\left(k^2-4\alpha^2\right)^2+\left(\frac{n_0+n_1}{n_2}\right)\alpha^2 \succ 15\alpha^4$ such that for $\left(k^2-4\alpha^2\right)^2+\left(\frac{n_0+n_1}{n_2}\right)\alpha^2 \prec 15\alpha^4$ the dispersion relation is not valid and the modulational instability occurs. In this condition, equation (36) is written

$$\omega = \pm i\sqrt{\frac{n_2}{n_0}\left[15\alpha^4-\left(k^2-4\alpha^2\right)^2-\left(\frac{n_0+n_1}{n_2}\right)\alpha^2\right]},\ \text{with}\ i^2 = -1. \tag{37}$$

The spectrum gain in this case is given by

$$G(\alpha,k)=\sqrt{\frac{n_2}{n_0}\left[15\alpha^4-\left(k^2-4\alpha^2\right)^2-\left(\frac{n_0+n_1}{n_2}\right)\alpha^2\right]}. \quad (38)$$

- If $n_0 n_2 \succ 0$ and $(n_0+n_1)n_2 \prec 0$, the dispersion relation (36) is valid for $\left(k^2-4\alpha^2\right)^2 \succ -\left(\frac{n_0+n_1}{n_2}\right)\alpha^2+15\alpha^4$ such that for $\left(k^2-4\alpha^2\right)^2 \prec -\left(\frac{n_0+n_1}{n_2}\right)\alpha^2+15\alpha^4$ the dispersion relation is not valid and the modulational instability occurs. In this condition, equation (36) is written

$$\omega=\pm i\sqrt{\frac{n_2}{n_0}\left[-\left(k^2-4\alpha^2\right)^2-\left(\frac{n_0+n_1}{n_2}\right)\alpha^2+15\alpha^4\right]},\ \text{with } i^2=-1; \quad (39)$$

Such that we can define the spectrum gain as

$$G'(\alpha,k)=\sqrt{\frac{n_2}{n_0}\left[-\left(k^2-4\alpha^2\right)^2-\left(\frac{n_0+n_1}{n_2}\right)\alpha^2+15\alpha^4\right]}. \quad (40)$$

In the case where $n_0 n_2 \prec 0$, equation (36) becomes equation (39) and vice versa.

5. Conclusions

Our aim was to construct solutions of shape $a\cosh^n(\alpha x-\alpha_0 t)$ of Boussinesq equation and to study the modulational instability of these solutions By the Bogning – Djeumen Tchaho Kofané method. To attain our objective, we define the generalized Boussinesq equation by changing the knowing coefficients by the unknown coefficients such that the equation can be written $n_0U_{tt}+n_1U_{xx}+n_2U_{xxxx}+n_3\left(U^3\right)_{xx}=0$. This equation will call the generalized Boussinesq equation. So looking for the general solution of the generalized Boussinesq equation as indicated previously, we see that this type of solution exists only for $n=1$ and according to the values of coefficients n_0, n_1, n_2 and n_3. After the general resolution of the generalized Boussinesq equation, we deduce from the resulting solutions the solution of the classical Boussinesq equation by setting $n_0=1$, $n_1=-1$, $n_2=-1$ and $n_3=-3$. The report we make is that we can obtain many modified Boussinesq equations which admit the solution of the type $U(x,t)=a\sec h(\alpha x-\alpha_0 t)$ by choosing adequately the values of n_0, n_1, n_2 and n_3. We also investigate the modulational instability of such solitary wave solution in the generalized Boussinesq equation. We found out the general condition for which this modulational instability is possible. Finally, we deduce from the general criteria the criteria of modulational instability in the classical Boussinesq equation as given by equation (1).

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