

# Implicit Hybrid Block Numerov-Type Method for the Direct Solution of Fourth-Order Ordinary Differential Equations

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**Abstract** This article proposes a new class of Implicit Hybrid Block Numerov-type methods for the direct solution of fourth order initial value problems of ordinary differential equations. This was achieved by constructing a continuous representation of Implicit hybrid Numerov-type schemes via interpolation and collocation of the approximate and derivative function respectively. The main discrete scheme was generated and new additional schemes were evaluated, which are combined and applied in block form as simultaneous numerical integrators. Of great interest are the basic properties of the new method such as, convergence, order, error constant and zero-stability. These basic properties were investigated. The performances of the methods were demonstrated on some numerical examples to show accuracy and efficiency advantages. The results compare favourably with the results when the same schemes are implemented in predictor-corrector mode.

**Keywords** Numerov-type methods, Implicit Hybrid Scheme, Ordinary Differential Equations (ODEs), Initial Value Problems (IVPs)

## 1. Introduction

Consider the  $n$ th-order initial value problem of ordinary differential equation of the form;

$$y^n = f(x, y(x), y'(x), y''(x), \dots, y^{n-1}(x)), \\ + y(a) = y_0, \dots, y^i(a) = y_i, i = 1(1)n - 1, \quad (1)$$

where  $f \in R$  is sufficiently differentiable and satisfies a Lipschitz condition. This class of problems has a wide variety of applications in science and engineering field especially in mechanical systems without dissipation, control theory and celestial mechanics (see Laudau and Liftshitz (1989); Liboff (1980)). However, only a limited number of analytical methods are available for solving (1) directly without reducing it to a first order system of differential equations. There exists an extensive literature on numerical methods for the solution of ordinary differential equations (see, for example Lambert (1973, 1976, 1991), Adey *et al.*, (2005), Awoyemi and Idowu (2005), Twizell and Khaliq (1984) and Sirisena *et al.*, (2004)). In particular, Awoyemi and Idowu (2005) developed a class of hybrid collocation methods for third-order ordinary differential equations. Sirisena *et al.*, (2004) constructed a Butcher-type two-step hybrid multistep method for accurate and efficient parallel solution of ordinary differential equations.

Hybrid methods were first proposed to overcome the Dahlquist (1956) barrier theorem whereby the conventional linear multistep method was modified to incorporating off-step points of derivation process (see Lambert (1973) and Gear (1965)). They utilize data at other points, other than the step points  $x_{n+j} = x_n + jh$ . Hybrid methods increase the order of the method while preserving good stability properties.

Thompson and Gonzalez (1997) described Numerov's method as an efficient and one of the most widely used algorithms for solving special second order linear ordinary differential equations of the form  $y'' = f(x, y)$ . He gave the following examples; one-dimensional time-independent Schrödinger equation, the equation of motion of an undamped forced harmonic oscillator and Poisson's equations.

Moreover, according to Vigo-Aguiar and Ramos (2005), different authors have dealt with  $y'' = f(x, y)$  in providing different approaches to solving it, but the pioneer work was probably due to Stömer (Lambert (1973)), who developed his method in connection with numerical calculations concerning the aurora borealis. The  $k$ -step Stömer method may be derived similarly to the Adams method, by twice integrating the differential equation and then replacing  $f$  by the interpolating polynomial passing through the points  $(x_{n-(k-1)}, y_{n-(k-1)}), \dots, (x_n, y_n)$ , where the  $x_i$  are equally spaced. However, to obtain more accurate formulas, it is possible used the interpolation polynomial passing through the additional points  $(x_{n+1}, y_{n+1})$  in which implicit Stömer method was obtained (also known in certain

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contexts as the Cowell method or the Numerov method (Vigo-Aguiar and Ramos (2005)). In Hans Van de Vyver (2007), an explicit Numerov-type method for second-order differential equations with oscillating solution was proposed.

All authors mentioned above independently developed Numerov's methods for the solution of special second order ordinary differential equations but little discussion was devoted to the modification of Numerov's method to handle general higher order ordinary differential equations.

In the research work of Awari *et al.*, (2013), the application of two-step continuous hybrid Butcher's method in block form for the solution of first order initial value problem was considered. Akinfenwa *et al.*, (2011) derived implicit two step continuous hybrid block methods with four off-steps points for solving stiff ordinary differential equation.

Therefore, this article proposes a class of continuous implicit hybrid block Numerov- type method for the direct solution of fourth order ordinary differential equations directly without reducing to system of first-order ordinary differential equations. The block methods can be seen as a set of linear multistep methods simultaneously applied to (1) and then combined to yield an approximation with better accuracy, better stability, better efficiency in parallel computing.  $\lambda$

**Definition 1.1:** The block method is said to be zero-stable if the roots  $\lambda_j$ ,  $j = 1, 2, \dots, s$  of the characteristic polynomial  $\rho(\lambda)$  defined by  $\rho(\lambda) = \det[\sum_{i=0}^s A^i \lambda^{s-i}] = 0$  satisfies  $|\lambda_j| \leq 1$  and for those roots with  $|\lambda_j| = 1$ , the multiplicity must not exceed the order of the differential equation. (see Fatunla (1994)).

**Definition 1.2:** The set of  $W$  equals  $\tau \in \mathbb{C}$ ; all roots  $\xi_i(\tau)$  of the characteristic equation satisfy  $|\xi_i(\tau)|$ , multiple roots satisfy  $|\xi_i(\tau)| < 1$ . is called the stability region or region of absolute stability of the method (Hairer and Wanner (1996)).

**Definition 1.3:** The LMM is said to be consistent if it has order  $P \geq 1$ .

The paper is organized as follows. Section 1 is of an introductory nature. The construction and implementation of the implicit hybrid block Numerov -type methods were described in Section 2. Stability analysis of the new methods was discussed in Section 3. In Section 4, some numerical experiments and results showing the relevance of the new methods are discussed. Finally, in Section 5 some conclusions are drawn.

## 2. Construction of the Implicit Hybrid Block Numerov-Type Methods

Consider the fourth-order initial value problem:

$$y^{iv} = f(x, y, y', y'', y'''), y(a) = \eta_0,$$

$$y'(a) = \eta_1, y''(a) = \eta_2, y'''(a) = \eta_3 \quad (2)$$

where  $f$  is sufficiently smooth,  $f: \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m$ ,  $y$  is an  $m$ -dimensional vector and  $x$  is a scalar variable and a set of equally spaced points on the integration interval also given by

$$a = x_0 < x_1 < \dots < x_n < \dots < x_{n+k} < x_N = b$$

with a specified positive integer step number  $k$  greater than zero. In this work,  $h$  is assumed to be a constant step-size given by  $h = x_{n+1} - x_n$ ,  $n = 1, \dots, N$ ;  $hN = b - a$ .

Assuming an approximate solution to (2) in form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \quad (3)$$

where  $a_j \in \mathbb{R}$ ,  $j = 0(1)r + s - 1$ ,  $y \in C^m(a, b) \subset P(x)$ ,  $r$  is the interpolation points and  $s$  is the collocation points. The highest derivative of (3) is given as:

$$y^{iv}(x) = \sum_{j=0}^{r+s-1} j(j-1)(j-2)\dots(j-4)a_j x^{j-4} \quad (4)$$

From (2) and (4), one obtains

$$f(x, y, y', y'', y''') = \sum_{j=0}^{r+s-1} j(j-1)(j-2)(j-3)a_j x^{j-4} \quad (5)$$

For one-step method ( $k=1$ ), we collocate (5) at  $x = x_{n+j}$ ,  $j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$  and interpolate (3) at  $x = x_{n+j}$ ,

$j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$  lead to system of equations with parameters  $a_j$ ,  $j = 0, 1, \dots, 8$ . Solving for these parameters and substituting into (3) using Maple software package, yields a linear multistep method with continuous coefficients of the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^4 \left( \sum_{j=0}^k \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \quad (6)$$

Using the transformation  $t = \frac{x - x_n}{h}$ ,  $\frac{dt}{dx} = \frac{1}{h}$ .

The coefficients of  $y_{n+j}$  and  $f_{n+j}$  are obtained as:

$$\alpha_0(t) = \frac{-16}{3}t^3 + 12t^2 - \frac{20}{3}t + 1$$

$$\alpha_{1/4}(t) = \frac{16}{5}(4t^3 - 8t^2 + 3t)$$

$$\alpha_{1/2}(t) = -(8t^3 - 14t^2 + 3t)$$

$$\alpha_{3/2}(t) = \frac{1}{15}(8t^3 - 6t^2 + t)$$

$$\beta_0(t) = \frac{h^4}{3870720} [12288t^8 - 79872t^7 + 200704t^6 - 247296t^5 + 161280t^4 - 59556t^3 + 12296t^2 - 1083]$$

$$\beta_{\frac{1}{4}}(t) = \frac{-h^4}{120960} [12288t^8 - 73728t^7 + 157696t^6 - 129024t^5 + 52836t^3 - 24328t^2 + 3147t]$$

$$\beta_{\frac{1}{2}}(t) = \frac{h^4}{1290240} [12288t^8 - 67584t^7 + 121856t^6 - 64512t^5 - 35364t^3 + 30664t^2 - 5307t]$$

$$\beta_{\frac{3}{2}}(t) = \frac{h^4}{19353600} [12288t^8 - 43008t^7 + 50176t^6 - 21504t^5 + 4956t^3 - 2488t^2 + 357t]$$

$$\beta_1(t) = \frac{h^4}{3870720} [12288t^8 - 55296t^7 + 71680t^6 - 32256t^5 + 29148t^3 - 19960t^2 + 3237t] \quad (7)$$

Evaluating (6) at  $t = 1 \Rightarrow x = x_{n+2}$  gives the discrete scheme:

$$y_{n+1} - \frac{1}{5}y_{n+\frac{3}{2}} - 3y_{n+\frac{1}{2}} + \frac{16}{5}y_{n+\frac{1}{4}} - y_n = \frac{h^4}{921600} [-259f_n + 848f_{n+\frac{1}{4}} - 5685f_{n+\frac{1}{2}} + 37f_{n+\frac{3}{2}} - 2105f_{n+1}] \quad (8)$$

The discrete scheme (8) is consistent and of order 5 with error constant  $C_9 = -1.06296e^{-4}$ .

The first, second and third derivatives of the continuous scheme (7) are found and evaluated at  $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{2}, 1$ .

Express explicitly as:

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{1}{384}h^3y'''_n + h^4\left(\frac{1385}{12386304}f_n + \frac{1469}{19353600}f_{n+\frac{1}{4}} - \frac{17}{589824}f_{n+\frac{1}{2}} - \frac{179}{309657600}f_{n+\frac{3}{2}} + \frac{55}{12386304}f_{n+1}\right) \quad (9)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + h^4\left(\frac{307}{241920}f_n + \frac{131}{75600}f_{n+\frac{1}{4}} - \frac{37}{80640}f_{n+\frac{1}{2}} - \frac{11}{1209600}f_{n+\frac{3}{2}} + \frac{17}{241920}f_{n+1}\right) \quad (10)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n + h^4\left(\frac{81}{1792}f_n + \frac{297}{2800}f_{n+\frac{1}{4}} + \frac{81}{1792}f_{n+\frac{1}{2}} - \frac{27}{44800}f_{n+\frac{3}{2}} + \frac{27}{1792}f_{n+1}\right) \quad (11)$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\frac{23}{1890}f_n + \frac{128}{4725}f_{n+\frac{1}{4}} + \frac{1}{630}f_{n+\frac{1}{2}} - \frac{1}{9450}f_{n+\frac{3}{2}} + \frac{1}{1080}f_{n+1}\right) \quad (12)$$

The implementation of this work was done in block mode in which the main method is constructed and additional schemes were evaluated, which are combined as a block and applied simultaneously process to solve (2).

However, following Fatunla (1991, 1994), the general discrete block formula is given as:

$$A^0 Y_m = ey_n + h^\mu dF(Y_m) + h^\mu BF(Y_m) \quad (13)$$

where e, d are vectors, B are RxR matrix and  $A^0$  identity matrix,  $\mu$  is the order of differential equation and

$$B = \begin{bmatrix} \frac{1469}{19353600} & -\frac{17}{589824} & -\frac{179}{309657600} & \frac{55}{12386304} \\ \frac{131}{75600} & -\frac{37}{80640} & -\frac{11}{1209600} & \frac{17}{241920} \\ \frac{296}{2800} & \frac{81}{1792} & -\frac{27}{44800} & \frac{27}{1792} \\ \frac{128}{4725} & \frac{1}{630} & -\frac{1}{9450} & \frac{1}{1080} \end{bmatrix} \quad (14)$$

$$d = \left[ \frac{1385}{12386304}, \frac{307}{241920}, \frac{81}{1792}, \frac{23}{1890} \right]^T \quad (15)$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (16)$$

$$Y_m = [y_{n+1/4}, y_{n+1/2}, y_{n+3/2}, y_{n+1}], \quad (17)$$

$$F(Y_m) = [f_{n+1/4}, f_{n+1/2}, f_{n+3/2}, f_{n+1}]. \quad (18)$$

They are uniformly of order  $[5, 5, 5, 5]^T$  with error constant

$$C_9 = \left[ \frac{11}{1698693120}, \frac{187}{1857945600}, \frac{27}{4587520}, \frac{1}{907200} \right]^T \quad (19)$$

**Comment:** It is worth mentioning that all the discrete schemes arising from a single continuous formula and its derivatives functions when evaluate at  $t \in (0, 2)$  are independent and infinite are of uniform order of accuracy.

Similarly, for **two-step method** ( $k = 2$ ), equation (5) is collocated at  $x = x_{n+j}, j = 0\left(\frac{1}{2}\right)k$  and equation (3) is interpolated at  $x = x_{n+j}, j = 0\left(\frac{1}{2}\right)\frac{3}{2}$  leads to a system of equations which after some manipulations yields the continuous method expressed in the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^4 \left( \left( \sum_{j=0}^k \beta_j(x) f_{n+j} \right) + \beta_u(x) f_{n+u} + \beta_v(x) f_{n+v} \right) \quad (20)$$

where  $u = \frac{1}{2}, v = \frac{3}{2}$ .

The coefficients  $\alpha_j(x)$  and  $\beta_j(x)$  are expressed as functions of  $t$ .

$$\alpha_0(t) = \frac{-1}{3}(-t + 4t^3)$$

$$\alpha_{1/2}(t) = 2(-t + t^2 + 2t^3)$$

$$\alpha_1(t) = -(-1 - t + 4t^2 + 4t^3)$$

$$\alpha_{3/2}(t) = \frac{2}{3}(t + 3t^2 + 2t^3)$$

$$\beta_0(t) = \frac{h^4}{483840} [55t + 11t^2 - 364t^3 + 672t^5 - 224t^6 - 384t^7 + 192t^8]$$

$$\beta_{\frac{1}{2}}(t) = \frac{-h^4}{120960} [493t + 53t^2 - 2296t^3 + 1344t^5 - 896t^6 - 192t^7 + 192t^8]$$

$$\beta_1(t) = \frac{h^4}{80640} [-553t - 773t^2 + 2212t^3 + 3360t^4 - 1120t^6 + 192t^8]$$

$$\beta_{\frac{3}{2}}(t) = \frac{-h^4}{120960} [-59t + 53t^2 + 560t^3 - 1344t^5 - 896t^6 + 192t^7 + 192t^8]$$

$$\beta_2(t) = \frac{h^4}{483840} [-41t + 11t^2 + 308t^3 - 672t^5 - 224t^6 + 384t^7 + 192t^8] \quad (21)$$

Evaluating (21) at  $t=1 \Rightarrow x = x_{n+2}$  gives the discrete scheme

$$y_{n+2} - 4y_{n+1/2} + 6y_{n+1} - 4y_{n+3/2} + y_n = \frac{h^4}{11520} [-f_n + 124f_{n+1/2} + 474f_{n+1} + 124f_{n+3/2} - f_{n+2}] \quad (22)$$

The discrete scheme (22) is consistent and of order 6 with error constant  $C_{10} = \frac{1}{3096576} = 3.229373347 \times 10^{-7}$ .

The first, second and third derivatives of the continuous scheme (21) are found and evaluated at  $t = 0\left(\frac{1}{2}\right)2$  and after some manipulations gives the explicit discrete block schemes:

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + h^4\left(\frac{3373}{1935360}f_n + \frac{139}{96768}f_{n+\frac{1}{2}} - \frac{283}{322560}f_{n+1} + \frac{179}{483840}f_{n+\frac{3}{2}} - \frac{131}{1935360}f_{n+2}\right) \quad (23)$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\frac{37}{1890}f_n + \frac{59}{1890}f_{n+\frac{1}{2}} - \frac{1}{72}f_{n+1} + \frac{11}{1890}f_{n+\frac{3}{2}} - \frac{1}{945}f_{n+2}\right) \quad (24)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n + h^4\left(\frac{5319}{71680}f_n + \frac{2889}{17920}f_{n+\frac{1}{2}} - \frac{1539}{35840}f_{n+1} + \frac{81}{3584}f_{n+\frac{3}{2}} - \frac{297}{71680}f_{n+2}\right) \quad (25)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4\left(\frac{176}{945}f_n + \frac{64}{135}f_{n+\frac{1}{2}} - \frac{16}{315}f_{n+1} + \frac{64}{945}f_{n+\frac{3}{2}} - \frac{2}{189}f_{n+2}\right) \quad (26)$$

which are expressed in block form (13) where,

$$B = \begin{bmatrix} \frac{139}{96768} & -\frac{283}{322560} & \frac{179}{483840} & -\frac{131}{96768} \\ \frac{59}{1890} & -\frac{1}{72} & \frac{11}{1890} & -\frac{1}{945} \\ \frac{2889}{17920} & -\frac{1539}{35840} & \frac{81}{3584} & -\frac{297}{71680} \\ \frac{64}{135} & -\frac{16}{315} & \frac{64}{945} & -\frac{2}{189} \end{bmatrix} \quad (27)$$

$$d = \begin{bmatrix} \frac{3373}{1935360} & \frac{37}{1890} & \frac{5319}{71680} & \frac{176}{945} \end{bmatrix}^T$$

$$Y_m = \begin{bmatrix} y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2} \end{bmatrix}$$

$$F(Y_m) = \begin{bmatrix} f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2} \end{bmatrix}$$

The block discrete schemes are uniformly of order six that is  $[6,6,6,6]^T$  with error constant

$$C_{10} = \begin{bmatrix} \frac{3619321}{3715891200}, \frac{-2111309}{3628800}, \frac{2210409}{45875200}, \frac{-716}{14175} \end{bmatrix}^T \quad (31)$$

In the case **three-step method** ( $k = 3$ ), collocation is done at  $x = x_{n+j}, j = 0, 1, 2, \frac{5}{2}, 3$  and interpolation is done at  $x = x_{n+j}, j = 0, 1, 2, \frac{5}{2}$ . The continuous linear multistep method is given in the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^4 \left( \sum_{j=0}^k \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \quad (32)$$

where  $v = \frac{5}{2}$  and the coefficients  $\alpha_j(x)$  and  $\beta_j(x)$  are expressed as.

$$\begin{aligned} \alpha_0(t) &= \frac{1}{10}(t - t^2 - 2t^3) \\ \alpha_1(t) &= \frac{1}{3}(-2t + 3t^2 + 2t^3) \\ \alpha_2(t) &= \frac{-1}{2}(-2 + t + 5t^2 + 2t^3) \\ \alpha_{5/2}(t) &= \frac{8}{15}(2t + 3t^2 + t^3) \\ \beta_0(t) &= \frac{h^4}{2419200}[698t - 425t^2 - 2051t^3 + 672t^5 - 448t^6 - 96t^7 + 96t^8] \\ \beta_1(t) &= \frac{-h^4}{483840}[8570t - 7817t^2 - 18851t^3 + 1344t^5 - 1120t^6 + 96t^7 + 96t^8] \\ \beta_2(t) &= \frac{h^4}{161280}[-5510t + 2039t^2 + 15197t^3 + 6720t^4 - 2016t^5 - 896t^6 + 288t^7 + 96t^8] \\ \beta_{5/2}(t) &= \frac{-h^4}{151200}[-188t + 1465t^2 + 5299t^3 - 2688t^5 - 448t^6 + 384t^7 + 96t^8] \\ \beta_3(t) &= \frac{h^4}{483840}[-1222t + 919t^2 + 3325t^3 - 1344t^5 + 224t^6 + 480t^7 + 96t^8] \end{aligned} \quad (33)$$

Evaluating (33) gives the discrete scheme below

$$y_{n+3} + 3y_{n+2} - y_{n+1} - \frac{16}{5}y_{n+5/2} + \frac{1}{5}y_n = \frac{h^4}{57,600}[-37f_n + 2105f_{n+1} + 5685f_{n+2} - 848f_{n+5/2} + 295f_{n+3}] \quad (34)$$

The discrete scheme is of order 7 with error constant  $C_{11} = -\frac{847}{128000}$ . The main discrete scheme and the additional schemes obtained are writing explicitly as:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4 \left( \frac{2059}{75600}f_n + \frac{59}{2160}f_{n+1} - \frac{169}{5040}f_{n+2} + \frac{131}{4725}f_{n+5/2} - \frac{107}{15120}f_{n+3} \right) \quad (35)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4 \left( \frac{1432}{4725}f_n + \frac{536}{945}f_{n+1} - \frac{166}{315}f_{n+2} + \frac{2048}{4725}f_{n+5/2} - \frac{104}{945}f_{n+3} \right) \quad (36)$$

$$y_{n+5/2} = y_n + \frac{5}{2}hy'_n + \frac{25}{8}h^2y''_n + \frac{125}{48}h^3y'''_n + h^4 \left( \frac{491125}{774144}f_n + \frac{1084375}{774144}f_{n+1} - \frac{40625}{36864}f_{n+2} + \frac{45125}{48384}f_{n+5/2} - \frac{184375}{774144}f_{n+3} \right) \quad (37)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2y''_n + \frac{9}{2}h^3y'''_n + h^4 \left( \frac{459}{400}f_n + \frac{1593}{560}f_{n+1} - \frac{1053}{560}f_{n+2} + \frac{297}{175}f_{n+5/2} - \frac{243}{560}f_{n+3} \right) \quad (38)$$

Express in block form of (13) where

$$B = \begin{bmatrix} \frac{59}{2160} & \frac{-169}{5040} & \frac{131}{4725} & \frac{-107}{15120} \\ \frac{536}{945} & \frac{-166}{315} & \frac{2048}{4725} & \frac{-104}{945} \\ \frac{1084375}{774144} & \frac{-40625}{36864} & \frac{45125}{48384} & \frac{-184375}{774144} \\ \frac{1593}{560} & \frac{-1053}{560} & \frac{297}{175} & \frac{-243}{560} \end{bmatrix}, \quad (39)$$

$$d = \begin{bmatrix} 0 & 0 & 0 & \frac{2059}{75600} \\ 0 & 0 & 0 & \frac{1432}{4725} \\ 0 & 0 & 0 & \frac{491125}{774144} \\ 0 & 0 & 0 & \frac{459}{400} \end{bmatrix}, \quad (40)$$

$$Y_m = \begin{bmatrix} y_{n+1}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3} \end{bmatrix}, \quad (41)$$

$$F(Y_m) = \begin{bmatrix} f_{n+1}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3} \end{bmatrix} \quad (42)$$

The explicit discrete schemes in block form are uniformly of order seven that is  $[7, 7, 7, 7]^T$  with error constant

$$C_{11} = \left[ \frac{1759}{3628800}, \frac{-18112}{836325}, \frac{10626}{663552}, \frac{189}{6400} \right]^T \quad (43)$$

### 3. Stability Analysis

To investigate the stability properties of methods for solving the initial value problem (2), the block method (13) was normalized for easy analysis (see Fatunla (1994), Yusuf and Onumanyi (2002)). The single block for the one-step method, two-step method and three-step method have the same first characteristic polynomial of the form:

$$\begin{aligned} \rho(\lambda) &= \det [\lambda A^0 - A^1] \\ &= \det \left[ \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right] \end{aligned}$$

$$= \det \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\rho(\lambda) = \lambda^3(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = 1$$

This satisfies the condition for method (13) to be zero – stable. Hence the block method is zero stable (see definition (1.1), (1.2) and (1.3)). The block method is also consistent, as it has the order  $p$  greater than 1. Hence the convergence of the method is asserted as in the theorem 3.1 below.

#### Theorem 3.1: (Dahlquist 1956)

The necessary and sufficient conditions for a linear multistep to be convergent are that it be consistent and zero-stable.

#### 3.1. Region of Absolute Stability of the Method

The Locus method is used to determine the region of absolute stability (Lambert (1973)). The boundary locus method is given by

$$h(\theta) = \frac{\rho(r)}{\sigma(r)} = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

where  $\rho(r)$  and  $\sigma(r)$  are the first and second characteristics polynomial respectively as:

$$r = e^{i\theta} = \cos \theta + i \sin \theta$$

Using a Matlab program, the absolute stability region of the new methods is plotted below.

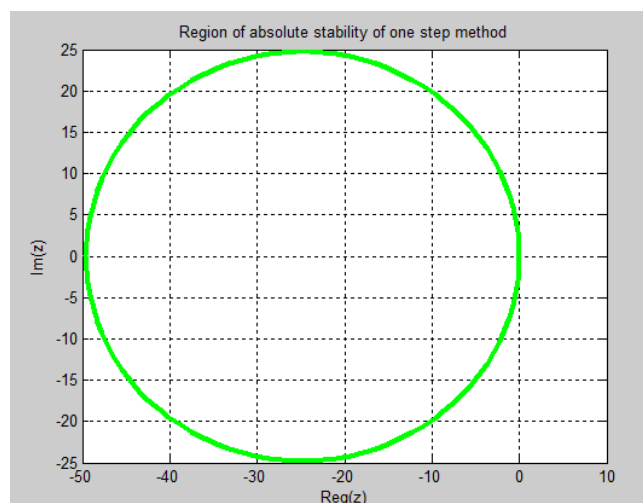
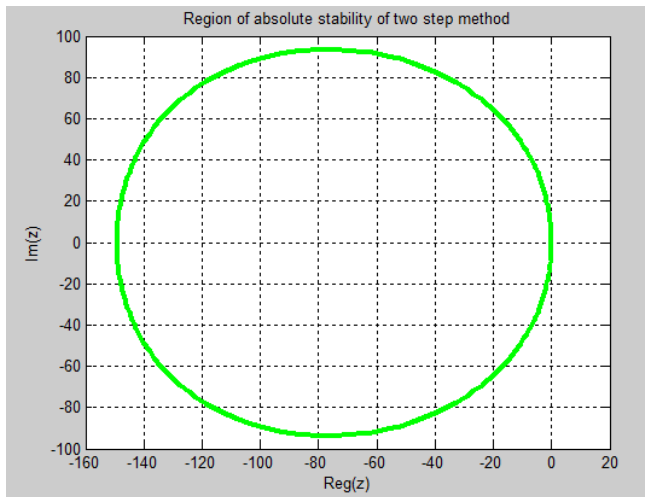
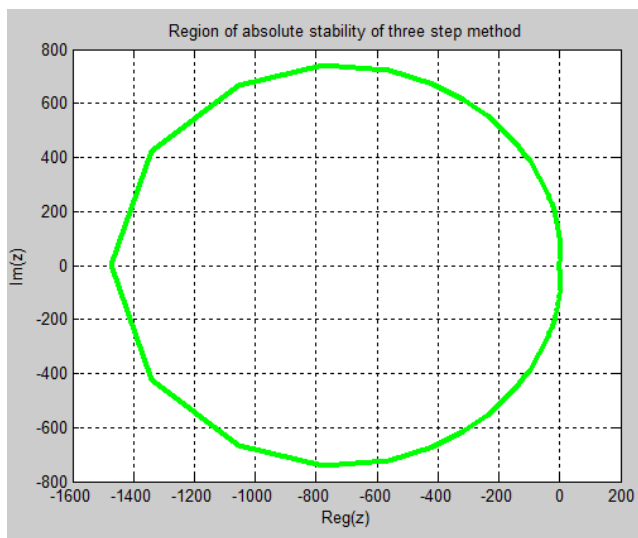


Figure 1. Region of Absolute Stability (RAS) of Block of One-step Method



**Figure 2.** Region of Absolute Stability (RAS) of Block of Two-step Method



**Figure 3.** Region of Absolute Stability (RAS) of the Block of Three-step Method

## 4. Numerical Experiments and Results

This section deals with the numerical experiments and

**Table 1.** The y-exact, approximate solution and error in One-step Numerov-type method for the non-linear problem 1

X	y-exact	y-approximate	Error in One-step Numerov-type method
<b>0.103125</b>	1.003139653527739149	1.003139653526590265	1.148884E-12
<b>0.206250</b>	1.006308634503762010	1.006308634484910542	1.8851468E-11
0.306250	1.009506973589071086	1.009506973491318106	9.7752980E-11
<b>0.406250</b>	1.012734701540634377	1.012734701224875248	3.15759129E-10
<b>0.506250</b>	1.015991849211685747	1.015991848424806972	7.86878775E-10
<b>0.603125</b>	1.019278447552026225	1.019278445888007693	1.66401853E-09
<b>0.703125</b>	1.022594527608326245	1.022594524466586058	3.14174019E-09
<b>0.803125</b>	1.025940120524428841	1.025940115065457591	5.45897125E-09
<b>0.903125</b>	1.029315257541653783	1.029315248639974225	8.90167956E-09
<b>1.003125</b>	1.032719969999102671	1.032719956193600508	1.38055022E-08

results. The derived schemes were tested on the following problems.

### Problem 1

Consider the non- linear fourth order ordinary differential equation

$$y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2)$$

Subject to the initial condition

$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, h = \frac{1}{320}.$$

### Problem 2

$$y^{(iv)} - 4y'' = 0, y(0) = 1, y'(0) = 3, y''(0) = 0,$$

$$y'''(0) = 16, 0 \leq x \leq 1.$$

The theoretical solution is given as

$$y(x) = 1 - x + e^{2x} - e^{-2x}$$

### Problem 3

$$y^{(iv)} = -\sin x + \cos x,$$

$$y'''(0) = 7, y''(0) = y'(0) = -1, y(0) = 0$$

Exact Solution

$$y(x) = -\sin x + \cos x + x^3 - 1.$$

### Problem 4

Consider the non-homogeneous linear equation of fourth order below

$$y^{iv} = \frac{-(8 + 25x + 30x^2 + 12x^3 + x^4)}{1 + x^2}$$

Subject to the initial conditions

$$y(0) = 0, y'(0) = 0, y''(0) = \frac{1}{144 - 100\pi},$$

$$y'''(0) = -3, h = \frac{1}{320}$$

The theoretical solution is  $y(x) = x^2 + e^x$



**Table 2.** The y-exact, approximate solution and error in 2-step method for the non-linear problem 1

X	y-exact	y-approximate	Error in 2-step Numerov-type method
<b>40.103125</b>	<b>1.003139653527739149</b>	<b>1.003139653526590265</b>	<b>1.148884E-12</b>
<b>0.206250</b>	1.006308634503762010	1.006308634484910542	1.8851468E-11
0.306250	1.009506973589071086	1.009506973491318106	9.7752980E-11
<b>0.406250</b>	1.012734701540634377	1.012734701224875248	3.15759129E-10
<b>0.506250</b>	1.015991849211685747	1.015991848424806972	7.86878775E-10
<b>0.603125</b>	1.019278447552026225	1.019278445888007693	1.66401853E-09
<b>0.703125</b>	1.022594527608326245	1.022594524466586058	3.14174019E-09
<b>0.803125</b>	1.025940120524428841	1.025940115065457591	5.45897125E-09
<b>0.903125</b>	1.029315257541653783	1.029315248639974225	8.90167956E-09
<b>1.003125</b>	1.032719969999102671	1.032719956193600508	1.38055022E-08

Tables 1 and 2 showed the y-exact, y-approximate and the error in one-step and two-step Numerov-type method for the non-linear problem 1.

It could be seen from tables 3, 4 and 5 that the maximum absolute error of the new Implicit Hybrid Block Numerov-type methods when they were implemented in block-mode are higher (more accurate) than those of maximum absolute error of the predictor -corrector mode for problems 2, 3 and 4.

**Table 3.** Accuracy Comparison of the 2-step Predictor-Corrector and 2-step Implicit Numerov-type method for linear problem 2

x	y-exact	y-approximate	Error in 2-step Predictor-Corrector	Error in 2-step Numerov-type method
<b>0.103125</b>	1.009375081380367279	1.009375081380367264	8.99999967E-10	1.500E-17
<b>0.206250</b>	1.018750651046752949	1.01875065104675280	1.896999989E-09	1.490E-16
0.306250	1.028127197304249133	1.028127197304248810	3.764500598E-09	3.230E-16
<b>0.406250</b>	1.037505208496096172	1.037505208496096066	5.730875704E-09	1.060E-16
<b>0.506250</b>	1.046885173022758589	1.046885173022758309	8.583461659E-09	2.800E-16
<b>0.603125</b>	1.056267579361003297	1.056267579361003713	1.153679635E-08	4.160E-16
<b>0.703125</b>	1.065652916082980786	1.065652916082981122	1.539206233E-08	3.360E-16
<b>0.803125</b>	1.075041671875310031	1.075041671875311603	1.934995149E-08	1.572E-15
<b>0.903125</b>	1.084434335558167877	1.084434335558169492	2.422550582E-08	1.615E-15
<b>1.003125</b>	1.093831396104383644	1.093831396104387025	2.920555833E-08	3.381E-15

**Table 4.** Accuracy Comparison of the 2-step Predictor-Corrector and 2-step Implicit Numerov-type method for problem 3

x	y-exact	y-approximate	Error in 2-step Predictor-Corrector	Error in 2-step Numerov-type method
<b>0.103125</b>	-0.003129847204687696	-0.0031298472046877018	9.8483400E-16	5.8350E-18
<b>0.206250</b>	-0.006269246355772101	-0.0062692463557721478	9.9997076E-13	4.6708E-17
0.306250	-0.009417983687528419	-0.0094179836875284719	9.0255603E-12	5.2467E-17
<b>0.406250</b>	-0.012575845339462482	-0.0125758453394625761	1.8673813E-11	9.3430E-17
<b>0.506250</b>	-0.015742617356611092	-0.015742617356611191	4.4663243E-11	9.9220E-17
<b>0.603125</b>	-0.018918085689843284	-0.0189180856898434241	7.0900122E-11	1.4019E-16
<b>0.703125</b>	-0.022102036196162510	-0.0221020361961626568	1.1882301E-10	1.4613E-16
<b>0.803125</b>	-0.025294254639009744	-0.0252942546390099310	1.6762209E-10	1.8712E-16
<b>0.903125</b>	-0.028494526688567489	-0.0284945266885676831	2.4347892E-10	1.9324E-16
<b>1.003125</b>	-0.003129847204687696	-0.0031298472046877018	9.8483400E-16	5.8350E-18

**Table 5.** Accuracy Comparison of the 3-step Predictor-Corrector and 3-step Implicit Numerov-type method for problem 4

X	y-exact	y-approximate	Error in 3-step Predictor-Corrector	Error in 3-step Numerov-type method
<b>0.103125</b>	1.009375081380367279	1.009375081380367264	9.059999511E-10	1.50000E-17
<b>0.206250</b>	1.018750651046752949	1.01875065104675280	1.914999829E-09	1.49000E-16
0.306250	1.028127197304249133	1.0281271973042480	2.820999533E-09	1.13300E-15
<b>0.406250</b>	1.037505208496096172	1.037505208496095127	4.711379410E-09	1.04500E-15
<b>0.506250</b>	1.046885173022758589	1.046885173022756815	6.707573030E-09	1.77400E-15
<b>0.603125</b>	1.056267579361003297	1.056267579361001485	8.597953171E-09	1.81200E-15
<b>0.703125</b>	1.065652916082980786	1.065652916082977641	1.149910769E-08	3.14500E-15
<b>0.803125</b>	1.075041671875310031	1.075041671875302866	1.450889367E-08	7.16500E-15
<b>0.903125</b>	1.084434335558167877	1.084434335558157753	1.741005130E-08	1.01240E-14
<b>1.003125</b>	1.093831396104383644	1.093831396104365728	2.134841320E-08	1.79160E-14

## 5. Conclusions

The Numerov's method which was initially designed for special second order initial value problems was modified in this paper to handle and solve general fourth order initial value problems of ordinary differential equations. The derived methods were implemented both in block and predictor-corrector mode. The block methods have the advantages of being self starting, are uniformly of the same order of accuracy and do not need predictors, having good accuracy as shown on numerical results of tables 3, 4 and 5. It should be noted that accuracy and efficiency rate of a method is dependent on the implementation strategies. If economical computation is required, then the new block methods are the better choice. The block method is recommended for general purposed use. Finally, the region of absolute stability of the block methods of one-step method, two-step method and three-step method were presented in figure 1, 2 and 3. Maple and Matlab software package were employed to generate the schemes and results.

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