

Solution of Stochastic Ordinary Differential Equations Using Explicit Stochastic Rational Runge-Kutta Schemes

M. R. Odekunle¹, M. O. Egwurube¹, K. A. Joshua^{2,*}

¹Department of Mathematics, Modibbo Adama University of Technology, Yola, Nigeria

²Department of Mathematics, Adamawa state University, Mubi, Nigeria

Abstract First order one-stage explicit Stochastic Rational Runge-Kutta methods were derived for the solution of stochastic ordinary differential equations. The derivation is based on the use of Taylor series expansion for both the deterministic and stochastic parts of the stochastic differential equation. The stability and convergence of the methods, found to be absolute stable. These methods were further tested on some numerical problems. From the results obtained, it is obvious that the derived methods; performed better than the ones with which we have analysed and they were compared with our results.

Keywords Stochastic differential equations, Runge-Kutta methods, Explicit rational Runge-Kutta methods

1. Introduction

Development in recent years in area of researches globally, in Mathematics and science related subjects such as, Engineering, Physics, Biology, Ecology, Hydrology, Economics, Investment, Finance, and population dynamics, just to mention but a few, have realised the importance of application of stochastic differential equations and the importance of random effects in most real problems. These are difficult to handle by the deterministic models [1, 2]. In view of this, there have been an increase in the need to construct stochastic models to simulate systems that deal with real life situations that contain uncertainties.

Many physical systems are modelled by stochastic differential equations where random effects (noise) are being modelled by a Brownian motion or what is called Wiener process [3]. Such differential equations are rarely solved analytically due to diffusion term involved. So numerical methods required and should be constructed in line with the principles of stochastic processes, to handle stochastic processes effects [4]. These models can offer more realistic representation of the physical system. Interesting enough, Runge-Kutta methods proves effective in handling stochastic differential equation theories that fits or handle stochastic processes, over some of the analytic methods [5, 6]. Therefore, there is a high need to develop and implement some stochastic Runge-Kutta methods for solving stochastic differential equations [7]. In this paper, an explicit Stochastic

Rational Runge-Kutta method is derived based on the modified approach of stochastic Runge-Kutta methods for solving stochastic ordinary differential equations.

Consider ordinary differential equations of the form,

$$y' = f(x, y), \quad y(a) = \eta, \quad f = \Re \times \Re \rightarrow \Re^m \quad (1)$$

The general Rational Runge-Kutta schemes for the solution of (1) according to [8, 9, 10, 11, 12] is expressed as:

$$y_{n+1} = \frac{y_n + h \sum_{i=1}^r c_i K_i}{1 + h y_n \sum_{i=1}^r v_i H_i} \quad (2)$$

where

$$K_i = f(x_n + c_i h, y_n + h \sum_{j=1}^r a_{i,j} K_j), \quad i = 1(1)r$$

$$H_i = g(x_n + d_i h, z_n + h \sum_{j=1}^r b_{i,j} H_j), \quad i = 1(1)r$$

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$$

$$\text{and } z_n = \frac{1}{y_n}$$

Consider the non-autonomous, one Wiener SDE of Stratonovich type:

$$dy(t) = f(t, y(t))dt + g(t, y(t)) \circ dW_t \quad (3)$$

* Corresponding author:

jakwanamu@yahoo.com (K. A. Joshua)

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$$Y(T) = Y(0) + \int_0^T f(t, y(t))dt + \int_0^T g(t, y(t)) \circ dW(t) \quad (4)$$

We shall assume a solution of the form

$$y_{n+1} = y_n + h \sum_{i=1}^s c_i K_i + J_i \sum_{i=1}^s s_i Ks_i \quad (5)$$

This now motivate us to formulate an s-stage explicit stochastic Rational Runge-Kutta (SRRK) methods for the solution of (3) as

$$y_{n+1} = \frac{y_n + h \sum_{i=1}^s c_i K_i}{1 + h \sum_{i=1}^s y_n v_i H_i} + J_l \left(\sum_{i=1}^s S_i Ks_i \right) \quad (6)$$

Where $h = \frac{t_{n+1} - t_n}{N}$, N is a positive integer,

$$J_l = \Delta W_n = \Delta W_{n+1} - \Delta W_n$$

$$K_i = f \left(t_n + h a_i, y_n + h \sum_{j=1}^s a_{ij} K_j \right) \text{ and } a_i = \sum_{j=1}^s a_{ij}$$

$$H_i = p(t_n, b_i h, y_n + h \sum_{j=1}^s b_{ij} H_j) \text{ and } b_i = \sum_{j=1}^s b_{ij}$$

$$p_n(t_n, z_n) = -z_n^2 f(t_n, y_n)$$

$$\text{and } z_n = \frac{1}{y_n}$$

$$Ks_i = g \left(t_n + h a_s i, y_n + J_l \sum_{j=1}^s b_{sij} Ks_j \right) \text{ and } a_s i = \sum_{j=1}^s b_{sij}$$

$$\sum_{i=1}^s (c_i + v_i) = 1$$

$$c_i, s_i, v_i, a_i, a_s i, a_{ij}, b_{ij}, \text{ and } b_{sij},$$

where $\text{for all } i, j = 1, 2, \dots, s$ are

constants to be determined. We can classify SRRK methods, as follows:

If $b_{ij} = a_{ij} = b_{sij} = 0$, $\forall i < j$, then the method is called semi-implicit.

If $b_{ij} = a_{ij} = b_{sij} = 0$, $\forall i \leq j$, then the method is called explicit.

Otherwise it is called implicit.

Definition 1: (Strong convergence)

We say that a discrete time approximation Y_c^h

converges strongly with order $p > 0$ at time T if there exist a positive constant C , which does not depend on the maximum step size h , and $\delta_0 > 0$, such that

$$E(\|Y_T - Y_c^h\|) \leq Ch^p$$

holds for each $h = \frac{T - t_0}{N} \in (0, \delta_0)$. N is the number of subintervals of the interval, $I = [t_0, T]$, Y_T is the exact solution at T , Y_c is the approximate solution at T . We shall employ the strong convergence principles in analysing our results.

2. Derivation of the Methods

In order to derive the one-stage explicit SRRK methods, let $s = 1$ in (6) to obtain

$$y_{n+1} = \frac{y_n + h c_1 K_1}{1 + h y_n v_1 H_1} + J_1 s_1 Ks_1 \quad (7)$$

where

$$K_1 = f(t_n, y_n)$$

$$H_1 = p(t_n, y_n)$$

$$Ks_1 = g(t_1, y_n)$$

$$\sum (c + v) = 1$$

$$h = \frac{t_{n+1} - t_n}{N} \quad N \text{ is a positive integer,}$$

$$J_1 = \Delta W_n = \Delta W_{n+1} - \Delta W_n$$

With

$$p(t_n, z_n) = -z_n^2 f(t_n, y_n)$$

$$z_n = \frac{1}{y_n}, \quad v_1 \neq 0$$

Expand the R.H.S of (7) binomially, simplify and neglect terms in h of order two and higher to obtain

$$y_{n+1} = y_n + h c_1 K_1 - h y_n^2 v_1 H_1 + J_1 s_1 Ks_1 \quad (8)$$

Expanding K_1 using Taylor series about (t_n, y_n) we have:

$$K_1 = f(t_n, y_n) + \dots \quad (9)$$

$$H_1 = p(t_n, y_n) + \dots \quad (10)$$

and

$$Ks_1 = g(t_n, y_n) + \dots \quad (11)$$

Substituting(9), (10) and (11) in the R.H.S of (8) we get

$$y_{n+1} = y_n + c_1 h f(t_n, y_n) - h y_n^2 v_1 p(t_n, z_n) + J_1 s_1 g(t_n, y_n) \quad (12)$$

But

$$p(t_n, z_n) = -z_n^2 f(t_n, y_n) \\ z_n = \frac{1}{y_n}, \quad v_1 \neq 0$$

Then (12) becomes

$$y_{n+1} = y_n + (c_1 + v_1) h f_n + \dots + J_1 s_1 g_n + \dots \quad (13)$$

From Taylor series expansion of y_{n+1} about x_n we get

$$y_{n+1} = y_n + h y' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y''' + \dots + J_1 y s' \\ + \frac{J_1^2}{2!} y s'' + \frac{J_1^3}{3!} y s''' + \dots \quad (14)$$

If we use the notations

$$y' = f_n \\ y'' = D f_n \quad (15)$$

since $y' = f(t, y)$, and let $y s' = g(t, y s)$ be the solution of the stochastic part denoted by $y s$ and when:

$$y'' = f_t + f f_y, \quad y s'' = g_t + g g_y$$

Therefore, (14) becomes

$$y_{n+1} = y_n + h f + \frac{h^2}{2!} (f_t + f f_y) + \dots \\ + J_1 g + \frac{J_1^2}{2!} (g_t + g g_y) + \dots \quad (16)$$

For consistency,

$$c_1 + v_1 = 1 \\ s_1 = 1 \quad (17)$$

Which is the simultaneous equations to be solved in three unknowns v_1, c_1, s_1 , where c_1 is a free parameter.

To determine a particular scheme of one-stage explicit Stochastic Rational Runge-Kutta Methods (SRRK):

Case 1, let

$$c_1 = \frac{1}{2}, \text{ then } v_1 = \frac{1}{2}, \text{ and } s_1 = 1$$

This gives

where

$$K_1 = f(t_n, y_n) \\ H_1 = p(t_n, z_n) \\ K s_1 = g(t_n, y_n) \\ \sum (c + v) = 1 \\ P(t_n, z_n) = -Z^2(t_n, y_n), \quad Z_n = \frac{1}{y_n}$$

and

$$h = \frac{t_{n+1} - t_n}{N}, \quad J_1 = \Delta W_n = \Delta W_{n+1} - \Delta W_n$$

Where J_1 is Gaussian random normal distribution with mean zero and standard deviation one ($N(0, 1)$)

Case 2, if

$$c_1 = \frac{1}{4}, v_1 = \frac{3}{4}, s_1 = 1, \text{ we obtain}$$

$$y_{n+1} = \frac{y_n + \frac{1}{4} h K_1}{1 + \frac{3}{4} h y_n H_1} + J_1 K s_1 \quad (19)$$

where $K_1, H_1, K s_1, P_n(t_n, z_n), h$, and J_1 are as defined in (18)

Case 3, if

$$c_1 = 0, v_1 = 1, s_1 = 1, \text{ then,}$$

$$y_{n+1} = \frac{y_n}{1 + h y_n H_1} + J_1 K s_1 \quad (20)$$

where $K_1, H_1, K s_1, P_n(t_n, z_n), h$, and J_1 are as defined in (18)

$$\text{Case 4, if } c_1 = \frac{2}{3}, v_1 = \frac{2}{3}, s_1 = 1$$

$$y_{n+1} = \frac{y_n + \frac{1}{3} h K_1}{1 + \frac{2}{3} h y_n H_1} + J_1 K s_1 \quad (21)$$

where $K_1, H_1, K s_1, P_n(t_n, z_n), h$, and J_1 are as

defined in (18)

Thus, the schemes (18), (19), (20) and (21) are the one-stage SRRK methods.

3. Stability Analysis

Theorem 1: (Convergence, [13])

- (i) Let the function $\varphi(x, y, h)$ be continuously jointly as a function of its three arguments, in the region \mathcal{F} defined by $X \in [a, b], y \in (-\infty, \infty), h \in [0, h_0]$ $h_0 > 0$
- (ii) Let $\varphi(h, y, h)$ satisfy a Lipchitz condition of the form $|\varphi(x, y^*, h) - \varphi(x, y, h)| \leq M |y^* - y|$ for all points $(x, y^*, h), (x, y, h)$ in \mathcal{F} then the method $y_{n+1} - y_n = h\varphi(x_n, y_n, h)$ is convergent if and only if it is consistent.

Theorem 2 [18]: For the test equation $dy = \lambda y dt + \sigma dW(t), y(0) = y_0, \lambda, \sigma \in C, R\lambda < 0$

the Euler-Maruyama method is numerical stable in mean if $|1 + \lambda h| < 1$.

Note that the stability condition $|1 + \lambda h| < 1$ is the same as Euler method for ODEs according to Lambert (1991). In general the numerical stability conditions for the additive case are coincident with them for ODEs (Hernandez and Spigler, 1992).

Theorem 3 [18]: If the Euler-Maruyama method satisfies the numerical stable condition in mean, i.e. $|1 + \lambda h| < 1$, the Euler-Maruyama method is asymptotically consistent in

mean square.

For the stability analysis of the derived schemes, we shall use the Shur method and utilize the principles of [14, 15, 16, 17, 18, 19], that states, the stability analysis of stochastic methods correspond with its deterministic counterpart. Using the test equation

$$y' = \lambda y \quad (22)$$

From (18)

$$y_{n+1} = (y_n + \frac{1}{2}hK_1)(1 + \frac{1}{2}hy_nH_1)^{-1}$$

Expanding and truncating terms in h of second and higher orders, we obtain

$$y_{n+1} = y_n + \frac{1}{2}hK_1 - \frac{1}{2}hy_n^2H_1 \quad (23)$$

But $y' = \lambda y$, and $K_1 = f(t_n, y_n) = f_n$ since $y' = f_n = \lambda y_n$

$$H_1 = p_n(t_n, z_n) = -z_n^2 f(t_n, y_n), \quad z_n = \frac{1}{y_n}$$

substituting these in (23), we get

$y_n + \lambda hy_n \Rightarrow (1 + \lambda h)y_n$ is the characteristic polynomial

$$|\zeta| = |1 + \lambda h| \leq 1, \quad \lambda < 0 \quad (24)$$

Therefore, $(-2.0, 0)$ is the interval of the absolute stability of the one-stage scheme (18). Similarly, we can see that (18), (19), (20) and (21) all have the same interval of absolute stability. Which is called numerical stable in mean in stochastic sense.

4. Numerical Problems and Results

Problem 1

Consider the SDE [6]

$$dy(t) = -(1 + 0.01y^2)(1 - y^2)dt + 0.1(1 - y^2)dW(t) \quad y(0) = 0$$

With the exact solution given by:

$$y(t) = \frac{(1 + y(0))e^{(-2t + 0.2W(t))} + y(0) - 1}{(1 + y(0))e^{(-2t + 0.2W(t))} - y(0) + 1}$$

Problem 2

Consider the SDE [20]

$$dy = -a^2 y(1 - y^2)dt + a(1 - y^2)dW(t), \quad y(0) = 0, \quad t \in [0, 1] \quad \text{with exact solution}$$

$$y(t) = \tanh(aW(t)) + \arctan h(y_0); \quad a = 1, \quad \varepsilon = 0.001$$

Therefore, the numerical solution using the explicit SRRK methods for the one-stage schemes as obtained in this work with absolute errors are given in the Tables 1 and 2 as seen below. The following notations will be used to represents results in the tables. PL, RAe1-3: Results from [6], Soheil, Results from [20] and JAK, Results obtained by our new methods.

Table 1. Numerical results of one-stage JAK explicit SRRK in comparison with [6] for Problem 1

t_i	W_i	Exact Solution	PL	Absolute Error	RAe1 (Pa)	Absolute error	RAe2	Absolute Error	RAE3	Absolute Error	JAK	Absolute Error
0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	-0.0439	-0.104	-0.1044	0.0004	-0.104	0	-0.1041	0.0001	-0.1038	0.0002	-0.102	0.0021
0.2	-0.0679	-0.2039	-0.2056	0.0017	-0.2038	0.0001	-0.204	0.0001	-0.2035	0.0005	-0.2024	0.0055
0.3	-0.0473	-0.2956	-0.2992	0.0035	-0.2952	0.0005	-0.2954	0.0002	-0.2947	0.0009	-0.2940	0.0001
0.4	-0.0951	-0.3881	-0.3943	0.0062	-0.3876	0.0005	-0.3878	0.0002	-0.387	0.0011	-0.3882	0.0001
0.5	-0.1686	-0.4753	-0.4846	0.0093	-0.4749	0.0004	-0.4752	0.0001	-0.4743	0.001	-0.4815	0.0010
0.6	0.0044	0.5367	-0.5474	0.0107	-0.5346	0.0021	-0.5349	0.0018	-0.5339	0.0028	-0.5367	0.0000
0.7	-0.0121	-0.6051	-0.6182	0.0131	-0.6028	0.0023	-0.6032	0.0019	-0.6022	0.003	-0.6031	0.0046
0.8	0.0556	-0.6609	-0.6754	0.0145	-0.6579	0.0031	-0.6582	0.0027	-0.6572	0.0037	-0.6593	0.0032
0.9	0.2192	-0.7055	-0.7204	0.015	-0.701	0.0044	-0.7014	0.0041	-0.7004	0.0051	-0.7073	0.0018
1.0	0.0809	-0.7582	-0.7748	0.0166	-0.7549	0.0033	-0.7552	0.003	-0.7543	0.0039	-0.7540	0.0000

Table 2. Numerical results of one-stage JAK explicit SRRK in comparison with [20] for Problem 2

	Soheili	JAK
h	absolute error	absolute error
0.0400	0.17316	0.00002
0.0200	0.10971	0.00001
0.0100	0.84066e ⁻¹	0.00000
0.0050	0.62554e ⁻¹	0.00000
0.0025	0.46519e ⁻¹	0.00000

convergence and accuracy, also the derived one -stage schemes was able to recover the results of their two stage schemes better. This means that our explicit SRRK methods is an alternative schemes to be used solve such problems. Also one is left with an options of choosing a scheme to work with in our case, unlike Euler Runge-Kutta method that has just a single scheme.

5. Discussion of Results

With the one- stage explicit stochastic rational Runge-Kutta schemes (SRRK) denoted JAK in the numerical results tables. We derived family of four schemes of one-stage SRRK methods The family schemes for the one-stage were tested to solve numerical Problems 1 and 2 from [6] and [20] respectively for some of the derived schemes as presented in tables 1 and 2. Matlab software (version 2010) was employed to run the simulations, based on normal distributed random numbers with mean zero and variance (standard deviation) one, i.e $N(0,1)$. From Tables 1 and 2 we can see the performance of our one stage schemes with the existing schemes in [6] and [20]. Also stability analysis of the one-stages developed were carried out using Schur method, in line with what we call mean and mean square stability principles in stochastic stability analysis discussed by some authors in section 3.0 and the stability analysis of the derived schemes showed they are bounded by the interval $(-2.0, 0)$, just as that of deterministic one-stage explicit Runge-Kutta methods.

6. Conclusions

Clearly our one- stage schemes performs better in terms of

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