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Periodic Oscillations of a Fourth Order Non Linear Ordinary Differential Equations

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Abstract This paper established the existence of periodic oscillations to the fourth order nonlinear ordinary differential equations of the form $x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex + \mu g(x) = \mu\delta(t)(*)\ by\ perturbation\ methods$. Also established is a periodicoscillation to equation (*) of the form $x(t)\ = Asin(\omega t + \varepsilon)\ +\ \mu Asin(3\omega t + \beta)\ +\ o(\mu)$ small 0, as $\mu \to 0\ where\ \ 0 \le \mu \ll 1$, is very small and A, ω, ε, b, c, d, e, and β are constants; and $\delta\ and\ g$ are continuous functions in their respective arguments that satisfy suitable growth restrictions such as $\frac{1}{T}\int_0^T \delta(s)ds = 0$ and $\frac{1}{T}\int_0^T g\big(x(s)\big)ds = 0$, $\delta(0) = 0$, $g(0) = 0$, $T = 2\pi\omega^{-1}$ is the least period of oscillations and ω is the angular frequency of oscillations.

Keywords Differential Equation, Perturbation Method, Periodic Oscillations, Continuous Functions, Angular Frequency

1. Introduction

Many physical problems arising from the motion electrical circuit theory and theory of elasticity can be represented by a fourth order ordinary differential equations of the form:

$$x^{iv} = f(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1}$$

Where $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dddot{x} = \frac{d^3x}{dt^3}$, $x^{iv} = \frac{d^4x}{dt^4}$, $\dddot{x}(t_0)\ = x_3,\ \ddot{x}(t_0) = x_2\ , \dot{x}(t_0) = x_1\ ,\ x(t_0) = x_0$ and the function f is continuous in all its argumentst, $x\ , \dot{x}, \ddot{x}\ , \dddot{x}$ respectively and has continuous partial derivatives with respect to $t\ , x\ , \dot{x}\ , \ddot{x}, \dddot{x}$ respectively.

By solution to equation (1) we mean a function $\varphi($ t $)$ that satisfy equation (1). That is the equation

$$\varphi^{iv}(t) = f(t, \varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), \dddot{\varphi}(t)) \tag{2}$$

is an identity. *An oscillatory solution* to equation (1) is a solution $\varphi(t)$ that has infinite number of zeros. The solution $\varphi(t)$ is a periodic solution if there exist a real number T such that

$$\varphi(t) = \varphi(t+T)\ \ , \forall\ t \in [0, \infty).$$

However, it is to be noted that

$$\varphi(0) = \varphi(T) = \varphi(2T) = \varphi(3T) = \tag{3}$$

The number T is called the least period of the periodic oscillations.

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Particularly in this paper we shall examine equation (1) when

$$f(t, x, \dot{x}, \ddot{x}, \dddot{x}) = -b\dddot{x} - c\ddot{x} - d\dot{x} - ex - \mu g(x) + \mu\delta(t) \tag{4}$$

Hence equation (1) becomes

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex + \mu g(x) = \mu\delta(t) \tag{5}$$

When $\mu = 0$, equation (5) becomes

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = 0 \tag{6}$$

The functions $\delta\ and\ g$ satisfy the following conditions:

$$\frac{1}{T}\int_0^T \delta(s)ds = 0, \tag{7}$$

$$\delta(0) = 0 \tag{8}$$

$$\frac{1}{T}\int_0^T g\big(x(s)\big)ds = 0 \tag{9}$$

$$xg(x) > 0 \tag{10}$$

$$g(0) = 0 \tag{11}$$

2. Objective of Research

The objectives of carrying out this research work are:

a. to apply the perturbation methods to the periodic oscillations of a fourth order ordinary differential equation (5) and
b. to bring out the relationship between the periodic oscillations of perturbed equation (5) and the periodic oscillations of the unperturbed equation (6).

These objectives shall be achieved by accepting the fact that the periodic oscillations of the unperturbed linear fourth order ordinary differential equation (6) of the form

$$x(t) = Asin(\omega t + \varepsilon) \quad (12)$$

can be extended to the periodic oscillations of the form

$$x(t) = Asin(\omega t + \varepsilon) + \mu Asin(3\omega t + \rho) + o(\mu) \quad (13)$$

$$smalloas\mu \to 0\ , \rho = 3\varepsilon$$

of perturbed fourth order nonlinear ordinary differential equation(5) due to the presence of additional correction term involving μ ($\mu Asin(3\omega t + \rho)$) in equation (13).

3. Problem Analysis

The first step is to consider a fourth order ordinary differential equations of the form

$$ax^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = 0 \quad (14)$$

The interest here is in a bounded periodic oscillation. Let $x(t) = e^{rt}$ be a solution to equation (14) where r is a constant. $x(t)$ will be a bounded periodic oscillation when t is very large if r is a complex number with negative real part. That is r = -m + I n, where m and n are positive real numbers and I ^{2} = - 1.

By substituting $x(t) = e^{rt}$ into the equation (9) we have the equation

$$ar^4 + br^3 + cr^2 + dr + e = 0 \quad (15)$$

where a, b, c, d, e are constant real numbers

Equation (15) is a quartic equation

Thus the equation (14) will have a bounded periodic oscillation if:

b = 0, d = 0, c ≥ 0 $and\ c^2 - 4ae < 0$ and since when $r^2 = k$ is a complex number, r is also a complex number.

Furthermore, it is imperative to consider another case of interest. Using the substitution.

$r = u - \frac{b}{4a}$ in equation(15) we have equation (15) becoming

$$\left(u - \frac{b}{4a}\right)^4 + \frac{b}{a}\left(u - \frac{b}{4a}\right)^3 + \frac{c}{a}\left(u - \frac{b}{4a}\right)^2 + \frac{d}{a}\left(u - \frac{b}{4a}\right) + \frac{e}{a} = 0 \quad (16)$$

Expanding the power of the binomials in equation (16) and simplifying, we have the follow in g equation

$$u^4 + \left\{-\frac{3b^2}{8a^2} + \frac{c}{a}\right\}u^2 + \left\{\frac{b^3}{8a^3} - \frac{bc}{2a^2} + \frac{d}{a}\right\}u + \left\{-\frac{3b^4}{256a^4} + \frac{cb^2}{16a^3} - \frac{bd}{4a^2} + \frac{e}{a}\right\} = 0 \quad (17)$$

Now let $h = -\frac{3b^2}{8a^4} + \frac{c}{a}$, $j = \frac{b^3}{8a^3} - \frac{bc}{2a^2} + \frac{d}{a}$, and $s = -\frac{3b^4}{256a^4} + \frac{cb^2}{16a^3} - \frac{bd}{4a^2} + \frac{e}{a}$

Equation (17) becomes

$$u^4 + hu^2 + ju + s = 0 \quad (18)$$

Equation (18) is a depressed quartic equation due to the absence of u^3 term in equation (18).

Case II. If j = 0we have a bi quadratic equation

$$u^4 + hu^2 + s = 0 \quad (19)$$

which is similar to what we have in case I. It is observed that if $u^2 = r$ is a complex number, r is also a complex number

4. Methods

The methods of perturbation was used to establish the existence of periodic oscillation to equation (1). We now consider the equation (14)

$$ax^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = 0$$

Without loss of generality we assume that a = 1, e = 1, d<0

The essence of d < 0 is to ensure that we have negative damping which will prevent the oscillation from being blown up after some time. We now assume a periodic oscillatory solution of the form

$$x(t) = psin\omega t + qcos\omega t \quad (20)$$

$$\dot{x}(t) = p\omega cos\omega t - q\omega sin\omega t \quad (21)$$

$$\ddot{x}(t) = -p\omega^2 sin\omega t - q\omega^2 cos\omega t \quad (22)$$

$$\dddot{x}(t) = -p\omega^3 cos\omega t + q\omega^3 sin\omega t \quad (23)$$

$$x^{iv}(t) = p\omega^4 sin\omega t + q\omega^4 cos\omega t \quad (24)$$

Substituting equations (20), (21), (22), (23), (24) into equation (14), we have the following equations

$$p\omega^4 sin\omega t + q\omega^4 cos\omega t - bp\omega^3 cos\omega t + bq\omega^3 sin\omega t\text{-}pc\omega^2 sin\omega t - qc\omega^2 cos\omega t$$

$$pd\omega cos\omega t + qd\omega sin\omega t + psin\omega t + qsin\omega t = 0 \quad (25)$$

Putting $\omega = 2\pi$, d = - 1 / 2, p=3 and q = 4 in equation (25) and after simplifying we have

$$b = \frac{-1}{8\pi^2} andc = 4\pi^2 + \frac{1}{4\pi^2} .$$

Hence the fourth order ordinary differential equation

$$x^{iv} - \left(\frac{1}{8\pi^2}\right)\dddot{x} + \left(4\pi^2 + \frac{1}{4\pi^2}\right)\ddot{x} - \frac{1}{2}\dot{x} + x = 0 \quad (26)$$

has the non-trivial periodic oscillation of the form

$$x(t) = 5sin(2\pi t + \varepsilon) \text{ where } \varepsilon = tan^{-1}(4/3)$$

5. Results

The results are as follows:

The following fourth order ordinary differential equations

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = 0 \quad (14)$$

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex + \mu x^3 = 0 \quad (27)$$

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex + \mu x^3 = \mu\delta(t) \quad (28)$$

$$x^{iv} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex + \mu g(x) = \mu\delta(t) \quad (29)$$

have periodic oscillations of the form

$$x(t) = Asin(\omega t + \varepsilon) \quad (30)$$

$$x(t) = Asin(\omega t + \varepsilon) + \mu Asin(3\omega t + \rho) \quad (31)$$

$$x(t) = Asin(\omega t + \varepsilon) + \mu Asin(3\omega t + \rho) + o(\mu) \quad (32)$$

$$x(t) = Asin(\omega t + \varepsilon) + \mu Asin(\omega t + \rho) + o(\mu) \quad (33)$$

Small 0 as $\mu \to 0$ *and* $\rho = 3\varepsilon$ respectively where the functions $\delta(t)$ *and* $g(x)$ satisfy the following conditions

$$\delta(0) = 0 \quad (35)$$

$$\delta(t) = \delta(t + T) \quad \forall\, t \in [0, \infty) \quad (36)$$

$$xg(x) > 0 \quad \forall x \approx> 0 \quad (37)$$

$$g(0) = 0 \quad (38)$$

$$\frac{1}{T}\int_T^0 g\big(x(s)\big)ds = 0 \quad (39)$$

6. Proof of Results

Clearly, the trivial solution $x(t) = 0$ is a periodic oscillation to equations (14), (27), (28) and (29). Since equation (14) is a constant coefficient fourth order ordinary differential equation, its periodic oscillations $x(t) = Asin(\omega t + \varepsilon)$ exist whenever b, c, d, e, ω satisfy the conditions $e = 1, d < 0, \omega = 2\pi$, c > 0

Assume $x(t) = Asin(\omega t + \varepsilon) + \mu Asin(3\omega t + 3\varepsilon)$ (40) *to be the solution of* equation (27). Clearly all the conditions of the statement of our results are satisfied. Hence the expression (40) is a periodic oscillation to equation (27).

Similarly there exist periodic oscillations of the form

$$x(t) = Asin(\omega t + \varepsilon) + \mu Asin(3\omega t + 3\varepsilon) + o(\mu) \quad (41)$$

smallo as $\mu \to 0$, to equations (28) and (29).

Also the zero solution $x(t) = 0$ is a periodic oscillation to equations (14), (27), (28) and (29).

7. Discussion

The periodic oscillation as we have in expression (30)

$$x(t) = Asin(\omega t + \varepsilon).$$

With A=5, $\omega = 2\pi, \varepsilon = tan^{-1}(4/3)$, obtained for the equation (14) when

$$b = \frac{-1}{8\pi^2}, c = 4\pi^2 + \frac{1}{4\pi^2} d = -\frac{1}{2}, e = 1$$

can be extended to the periodic oscillation of the nonlinear fourth order ordinary differential equations of the forms(27), (28) and (29), respectively, where

$$\mu isasmallparameterand\ 0 < \mu \ll 1.$$

It should be noted that equations (27), (28), and (29) reduces to equation (14) when $\mu = 0$. Also the periodic oscillations (31), (32) and (33) obtained for equations (27), (28) and (29) respectively reduces to the periodic oscillation we have in (30) obtained for equation (14) when $\mu = 0$ in equations (27), (28), (29) and expressions (31), (32) and (33).

8. Conclusions

It has been established that periodic oscillations apart from the trivial solution $x(t) = 0$ to equations (14), (27), (28), and (29) through systematic approach of constructing periodic oscillations by perturbation methods exist. Starting with the construction of a periodic oscillation (30) to a linear fourth order ordinary differential equation (14), we now extend this procedure to the construction of periodic oscillations to equations (27), (28), and (29) when $\mu \neq 0$. The results can be extended to a wider class of fourth order nonlinear ordinary differential equations which we shall consider in our future presentations.

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