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# New Twelfth Order Iterative Methods for Finding Multiple Roots of Nonlinear Equations

**R. Thukral**

Padé Research Centre, 39 Deanswood Hill, Leeds, West Yorkshire, LS17 5JS, England

**Abstract** The aims of this paper are, firstly, to present four new twelfth order iterative methods for solving nonlinear equations and secondly, to introduce new formulas for approximating the multiplicity of the iterative method. It is proved that these methods have the convergence order of twelve requiring six function evaluations per iteration. Numerical comparisons are included to demonstrate exceptional convergence speed of the proposed methods.

**Keywords** Modified Newton-type method, Root-finding, Nonlinear equations, Multiple roots, Order of convergence, Efficiency index

## 1. Introduction

Finding the root of nonlinear equations is one of important problem in science and engineering [1-28]. In this paper, we present four new multipoint higher-order iterative methods to find multiple roots of the nonlinear equation $f(x)=0$, where $f: I \subset \mathrm{R} \rightarrow \mathrm{R}$ for an open interval $I$ is a scalar function. The multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. In recent years, many modifications of the Newton-type methods for simple roots have been proposed and analysed [13] and little work has been done on multiple roots. Therefore, the prime motive of this study is to develop a new class of multi-step methods for finding multiple roots of nonlinear equations of a higher than the existing iterative methods. In order to construct the new twelfth order method for finding multiple roots we use the well-established fourth order method given in [15, 16, 17, 20]. The purpose of this paper is to show further development of the ninth order methods and introduce new formulas for approximating the multiplicity of the iterative methods. This paper is actually a continuation of the previous study [23]. The extension of this investigation is based on the improvement of the ninth order method. In addition, the new iterative methods have a better efficiency index than the eight to ten convergence order methods [10, 23]. Hence, the proposed twelfth order methods are significantly better when compared with these established methods.

* Corresponding author:
rthukral@hotmail.co.uk (R. Thukral)
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The structure of this paper is as follows. Some basic definitions relevant to the present work are presented in the section 2. In section 3 the new multi-point methods are defined and proved. In section 4 the new formulas for approximating the multiplicity of the iterative methods are described. In section 5, two well-established methods are stated, it will demonstrate the effectiveness of the new twelfth order iterative methods. Finally, in section 6, numerical comparisons are made to demonstrate the performance of the presented methods.

## 2. Preliminaries

In order to establish the order of convergence of the new twelfth order methods, we state some definitions:

**Definition 1** Let $f(x)$ be a real-valued function with a root $\alpha$ and let $\{x_n\}$ be a sequence of real numbers that converge towards $\alpha$. The order of convergence $p$ is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = \zeta \neq 0, \tag{1}$$

where $p \in \mathrm{R}^+$ and $\zeta$ is the asymptotic error constant [6, 13, 27].

**Definition 2** Let $e_k = x_k - \alpha$ be the error in the $k$th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + \mathrm{O}\left(e_k^{p+1}\right), \tag{2}$$

is the error equation. If the error equation exists, then $p$ is the order of convergence of the iterative method [6, 13, 27].

**Definition 3** Let $r$ be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$\sqrt[r]{p}, \tag{3}$$

where $p$ is the order of convergence of the method [6].

**Definition 4** Suppose that $x_{n-2}, x_{n-1}$ and $x_n$ are three successive iterations closer to the root $\alpha$. Then the computational order of convergence may be approximated by

$$\mathrm{COC} \approx \frac{\ln\left|\delta_n \div \delta_{n-1}\right|}{\ln\left|\delta_{n-1} \div \delta_{n-2}\right|}, \tag{4}$$

where $\delta_i = f\left(x_i\right) \div f'\left(x_i\right)$, [23].

## 3. Construction of the Methods and Convergence Analysis

In this section we define new twelfth order iterative methods for finding multiple roots of a nonlinear equation. In order to construct new twelfth order methods, we use well known fourth order iterative methods, presented by Thukral, Sharma et al., Shengguo et al. and Soleymani et al., [15, 16, 17, 20].

### 3.1. Method 1

It is well established that the first two step is the Thukral fourth order method [20] and the new third step is in the form of the Osada third order method [12]. Consequently, we obtain a new twelfth order method based on these two well-established method. The new scheme is given as

$$y_n = x_n - m\left(\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right), \tag{5}$$

$$z_n = y_n - m\left[\sum_{i=1}^{3} i\left(\frac{f\left(y_n\right)}{f\left(x_n\right)}\right)^{i/m}\right]\left(\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right), \tag{6}$$

$$x_{n+1} = z_n - \left(\frac{m}{2}\right)\left(m+1\right)\left(\frac{f\left(z_n\right)}{f'\left(z_n\right)}\right) + \left(\frac{1}{2}\right)\left(m-1\right)^2\left(\frac{f'\left(z_n\right)}{f''\left(z_n\right)}\right) \tag{7}$$

where $n \in \mathrm{N}$, $x_0$ is the initial guess and provided that the denominator of (7) is not equal to zero.

**Theorem 1**

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f: I \subset \mathrm{R} \to \mathrm{R}$ for an open interval $I$ with multiplicity $m$, which includes $x_0$ as an initial guess of $\alpha$. Then the iterative method defined by scheme (7) has twelfth order convergence.

**Proof**

Let $\alpha$ be a multiple root of multiplicity $m$ of a sufficiently differentiable function $f\left(x\right)$ and $f\left(\alpha\right) = 0$. We denote the errors given by each step as $e = x - \alpha$, $\tilde{e} = y - \alpha$ and $\hat{e} = z - \alpha$.

Using the Taylor series expansion of $f\left(x\right), f'\left(x\right), f\left(y\right), f'\left(y\right)$ about $\alpha$, we have

$$f(x_n) = \left(\frac{f^{(m)}\left(\alpha\right)}{m!}\right) e_n^m \left[1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right] \tag{8}$$

$$f'(x_n) = \left(\frac{f^{(m)}\left(\alpha\right)}{\left(m-1\right)!}\right) e_n^{m-1} \times \left[1 + \left(\frac{m+1}{m}\right) c_1 e_n + \left(\frac{m+2}{m}\right) c_2 e_n^2 + \cdots\right], \tag{9}$$

$$f(y_n) = \left(\frac{f^{(m)}\left(\alpha\right)}{m!}\right) \tilde{e}_n^m \left[1 + c_1 \tilde{e}_n + c_2 \tilde{e}_n^2 + c_3 \tilde{e}_n^3 + \cdots\right] \tag{10}$$

$$f'(y_n) = \left(\frac{f^{(m)}\left(\alpha\right)}{\left(m-1\right)!}\right) \tilde{e}_n^{m-1} \times \left[1 + \left(\frac{m+1}{m}\right) c_1 \tilde{e}_n + \left(\frac{m+2}{m}\right) c_2 \tilde{e}_n^2 + \cdots\right], \tag{11}$$

where $n \in \mathrm{N}$ and

$$c_k = \frac{m!\, f^{(m+k)}\left(\alpha\right)}{\left(m+k\right)!\, f^{(m)}\left(\alpha\right)}. \tag{12}$$

Moreover by (5), we have

$$y_n = e_n - m\frac{f\left(x_n\right)}{f'\left(x_n\right)} = e_n - e_n\left[1 - \frac{c_1}{m} e_n + \frac{\left(m+1\right)c_1^2 - 2mc_2}{m^2} e_n^2 + \cdots\right]. \tag{13}$$

The expansion of $f\left(y_n\right)$ about $\alpha$ and simplifying, yields

$$f(y_n) = \left(\frac{f^{(m)}\left(\alpha\right)}{m!}\right)\left(\frac{c_1}{m}\right)^m e_n^{2m} \times \left[1 + \left(\frac{2mc_2 - \left(m+1\right)c_1^2}{c_1}\right) e_n\right.$$

$$+\left(\frac{u_1 c_1^4 - 2m u_2 c_1^2 c_2 + 4m^2\left(m-1\right)c_2^2 + 6m^2 c_1 c_3}{2c_1^3}\right)e_n^2 + \cdots\Bigg] \tag{14}$$

where

$$u_1 = \left(m^3 + 3m^2 + 3m + 3\right), u_2 = \left(2m^2 + 3m + 2\right). \tag{15}$$

Dividing (14) by (8), we have

$$\frac{f(y_n)}{f(x_n)} = \left(\frac{c_1}{m}\right)^m e_n^m \left[1 + \left(\frac{\omega_1}{c_1}\right)e_n + \left(\frac{\omega_2}{2mc_1^2}\right)e_n^2 - \cdots\right]. \tag{16}$$

Furthermore, we have

$$\sqrt[m]{\frac{f(y_n)}{f(x_n)}} = \left(\frac{c_1}{m}\right)e_n + \left(\frac{\omega_1}{m^2}\right)e_n^2 + \left(\frac{\omega_2 - \left(m-1\right)\omega_1^2}{2m^3 c_1}\right)e_n^3 + \cdots \tag{17}$$

where

$$\left.\begin{aligned} \omega_1 &= 2mc_2 - \left(m+2\right)c_1^2, \\ \omega_2 &= \left(m+1\right)^2\left(m+3\right)c_1^4 - \\ &\quad 2m\left(m+1\right)\left(2m+3\right)c_1^2 c_2 + \\ &\quad 4m^2\left(m-1\right)c_2^2 + 6m^2 c_1 c_3. \end{aligned}\right\} \tag{18}$$

Substituting appropriate expressions in (6), we obtain

$$z_n - \alpha = y_n - \alpha - m\left[\sum_{i=1}^{3} i\left(\frac{f\left(y_n\right)}{f\left(x_n\right)}\right)^{i/m}\right]\left(\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right), \tag{19}$$

we obtain the asymptotic error constant

$$z - \alpha = 2^{-1}m^{-3}\left(mc_1^2 - 2mc_2 + 3c_1^2\right)e_n^4 + \cdots \tag{20}$$

We progress to expand $f\left(z\right), f'\left(z\right) f''\left(z\right)$ about $\alpha$, we have

$$f(z_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right)\hat{e}_n^m\left[1 + c_1\hat{e}_n + c_2\hat{e}_n^2 + c_3\hat{e}_n^3 + \cdots\right] \tag{21}$$

$$f'(z_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!}\right)\hat{e}_n^{m-1} \times \left[1 + \left(\frac{m+1}{m}\right)c_1\hat{e}_n + \left(\frac{m+2}{m}\right)c_2\hat{e}_n^2 + \cdots\right], \tag{22}$$

$$f''(z_n) = \left(\frac{f^{(m)}(\alpha)}{(m-2)!}\right)\hat{e}_n^{m-2} \times \left[1 + \left(\frac{m+1}{m-1}\right)c_1\hat{e}_n + \left(\frac{\left(m+1\right)\left(m+2\right)}{m\left(m-1\right)}\right)c_2\hat{e}_n^2 + \cdots\right] \tag{23}$$

Substituting appropriate expressions in (7),

$$e_{n+1} = z_n - \alpha - \left(\frac{m}{2}\right)\left(m+1\right)\left(\frac{f\left(z_n\right)}{f'\left(z_n\right)}\right) + \left(\frac{1}{2}\right)\left(m-1\right)^2\left(\frac{f'\left(z_n\right)}{f''\left(z_n\right)}\right). \tag{24}$$

Simplifying (24), we obtain the asymptotic error constant

$$e_{n+1} = 2^{-4}m^{-11}\left(m-1\right)^{-1}c_1^3 \times \left(m^2c_1^2 + 2mc_1^2 + 2mc_2 + c_1^2 - 2m^2c_2\right) \times \left(mc_1^2 - 2mc_2 + 3c_1^2\right)^3 e_n^{12}. \tag{25}$$

The expression (25) establishes the asymptotic error constant for the twelfth order of convergence for the new Newton-type method defined by (7).

### 3.2. Method 2

Another twelfth order iterative method is constructed by using a fourth order method presented by Shengguo et al. [15]. As before the first two steps is the Shengguo et al. method and third step is in the form of Osada third order method. The new twelfth-order iterative method is given as,

$$y_n = x_n - \left(\frac{2m}{m+2}\right)\left(\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right), \tag{26}$$

$$z_n = x_n - \left[\frac{\left(\frac{m}{2}\right)\left(m-2\right)\left(\frac{m}{m+2}\right)^{-m} f'\left(y_n\right) - \left(\frac{m^2}{2}\right)f'\left(x_n\right)}{f'\left(x_n\right) - \left(\frac{m}{m+2}\right)^{-m} f'\left(y_n\right)}\right] \times \left(\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right), \tag{27}$$

$$x_{n+1} = z_n - \left(\frac{m}{2}\right)(m+1)\left(\frac{f(z_n)}{f'(z_n)}\right) + \left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(z_n)}{f''(z_n)}\right). \tag{28}$$

**Theorem 2**

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f : I \subset \mathrm{R} \to \mathrm{R}$ for an open interval $I$ with multiplicity $m$, which includes $x_0$ as an initial guess of $\alpha$. Then the iterative method defined by scheme (28) has twelfth order convergence.

**Proof**

Using appropriate expressions in the proof of the theorem 1 and substituting them into (28), we obtain the asymptotic error constant

$$e_{n+1} = 54^{-1} m^{-14} (m-1)^{-1} (m+2)^{-6} \times \left(m^2 c_1^2 + 2mc_1^2 + 2mc_2 + c_1^2 - 2m^2 c_2\right) \times \left(s_1 c_1^3 - 3 s_2 c_1 c_2 + 3m^5 c_3\right)^3 e_n^{12} \tag{29}$$

where

$$s_1 = \left(m^3 + 2m^2 + 2m - 2\right)(m+2)^2, \tag{30}$$

$$s_2 = 3m^3 (m+2)^2, \tag{31}$$

The expression (29) establishes the asymptotic error constant for the twelfth order of convergence for the new Newton-type method defined by (28).

### 3.3. Method 3

The third twelfth order iterative method is based on the Sharma et al. fourth order method presented in [17]. Here also, the first two steps is the Sharma et al. method and third step is in the form of Osada third order method. The new twelfth order iterative method is given as,

$$y_n = x_n - \left(\frac{2m}{m+2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{32}$$

$$z_n = x_n - \left(\frac{m}{8}\right)\left[k_1\left(\frac{f(x_n)}{f'(x_n)}\right) - k_2\left\{(m-1) - k_2\left(\frac{f'(x_n)}{f'(y_n)}\right)\right\}\left(\frac{f(x_n)}{f'(y_n)}\right)\right] \tag{33}$$

$$x_{n+1} = z_n - \left(\frac{m}{2}\right)(m+1)\left(\frac{f(z_n)}{f'(z_n)}\right) + \left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(z_n)}{f''(z_n)}\right), \tag{34}$$

where

$$k_1 = \left(m^3 - 4m + 8\right), \tag{35}$$

$$k_2 = (m+2)^2\left(\frac{m}{m+2}\right)^m \times \left[\left(2(m-1) - (m+2)\left(\frac{m}{m+2}\right)^m\right)\right]. \tag{36}$$

**Theorem 3**

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f : I \subset \mathrm{R} \to \mathrm{R}$ for an open interval $I$ with multiplicity $m$, which includes $x_0$ as an initial guess of $\alpha$. Then the iterative method defined by scheme (34) has twelfth order convergence.

**Proof**

Using appropriate expressions in the proof of the theorem 1 and substituting them into (34), we obtain the asymptotic error constant

$$e_{n+1} = 54^{-1} m^{-14} (m-1)^{-1} (m+2)^{-6} \times \left(m^2 c_1^2 + 2mc_1^2 + 2mc_2 + c_1^2 - 2m^2 c_2\right) \times \left(s_3 c_1^3 - 3 s_4 c_1 c_2 + 3m^5 c_3\right)^3 e_n^{12}. \tag{37}$$

where

$$s_3 = \left(m^5 + 6m^4 + 14m^3 + 14m^2 + 12m - 8\right), \tag{38}$$

$$s_4 = 3m^3 (m+2)^2. \tag{39}$$

The expression (37) establishes the asymptotic error constant for the twelfth order of convergence for the new Newton-type method defined by (34).

### 3.4. Method 4

The fourth twelfth order iterative method is based on the Soleymani et al. fourth order method presented in [16]. Here also, the first two steps is the Soleymani et al. method and third step is in the form of Osada third order method. The new twelfth order iterative method is given as,

$$y_n = x_n - \left(\frac{2m}{m+2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{40}$$

$$z_n = x_n + \left[\frac{4mq_1{}^m t_1}{q_1{}^m t_2 - q_2 t_2}\right]\left[1 - q_2\left(t_2 - q_1{}^{m-1}\right)^2\right], \tag{41}$$

$$x_{n+1} = z_n - \frac{\left(\frac{m}{2}\right)(m+1)\left(\frac{f(z_n)}{f'(z_n)}\right) + }{\left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(z_n)}{f''(z_n)}\right)}, \tag{42}$$

where

$$q_1 = \left(\frac{m}{m+2}\right), \quad q_2 = \frac{m^3(m-2)}{16p^{2m}}, \tag{43}$$

$$t_1 = \frac{f(x_n)}{f'(x_n)}, \quad t_2 = \frac{f'(y_n)}{f'(x_n)}, \tag{44}$$

**Theorem 4**

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f: I \subset \mathrm{R} \to \mathrm{R}$ for an open interval $I$ with multiplicity $m$, which includes $x_0$ as an initial guess of $\alpha$. Then the iterative method defined by scheme (42) has twelfth order convergence.

**Proof**

Using appropriate expressions in the proof of the theorem 1 and substituting them into (42), we obtain the asymptotic error constant

$$e_{n+1} = 54^{-1} m^{-17} (m-1)^{-1} (m+2)^{-6} \times \left(m^2 c_1^2 + 2mc_1^2 + 2mc_2 + c_1^2 - 2m^2 c_2\right) \times \left(s_5 c_1^3 - 3s_6 c_1 c_2 + 3m^6 c_3\right)^3 e_n^{12} \tag{45}$$

where

$$s_5 = \left(m^4 + 2m^3 + 2m^2 - 8m + 12\right)(m+2)^2, \tag{46}$$

$$s_6 = 3m^4 (m+2)^2, \tag{47}$$

The expression (45) establishes the asymptotic error constant for the twelfth order of convergence for the new Newton-type method defined by (42).

# 4. New Formulas for Approximating the Multiplicity

In this section, we derive some new formulas to approximate the multiplicity $m$ of the method. In [26] Thukral presented a new formula for approximating the multiplicity $m$ as

$$\widehat{m}_n \approx \frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})}. \tag{48}$$

In fact this formula was discovered by Lagouanelle in [9] and apparently Thukral rediscovered this formula. However, empirically we have found that the formula should be expressed as

$$\widehat{m}_1 \approx \left|\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})}\right|, \tag{49}$$

The approximations obtained by the new and old formulas are based on the Schroder second order method [14], given as

$$x_n = x_{n-1} - \frac{f(x_{n-1}) f'(x_{n-1})}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})}. \tag{50}$$

**Definition 5** Suppose that $x_{n-1}, x_n$ and $x_{n+1}$ are three successive iterations closer to the root $\alpha$. Then the computational order of convergence may be approximated by the following;

$$\widehat{m}_2 \approx \frac{\ln|f(x_n)|}{\ln|f(x_n) \div f'(x_n)|}. \tag{51}$$

This formula was actually presented by Traub [27] and the following new formulas are actually the improvements of the above formulas;

$$\widehat{m}_3 \approx \left|(x_{n-1} - x_n) f'(x_{n-1}) f(x_{n-1})^{-1}\right|, \tag{52}$$

$$\widehat{m}_4 \approx \left|\frac{1 - r_1}{1 + r_1 - r_1 r_3^{-1}}\right|, \tag{53}$$

$$\widehat{m}_5 \approx \left|\frac{1 + r_1}{1 + r_1 - r_1 r_3^{-1}}\right|, \tag{54}$$

$$\widehat{m}_6 \approx \left|\frac{r_3 + r_2 + r_1^3}{r_3 - r_2 + r_1}\right|, \tag{55}$$

$$\widehat{m}_7 \approx \left|\frac{2r_1^{-1} + 3r_2 + 3r_3}{2r_1^{-1} - 2r_2 - 2r_3}\right|, \tag{56}$$

where

$$r_1 = \frac{f(x_n)}{f'(x_n)}, \quad r_2 = \frac{f(x_n)}{f''(x_n)}, \quad r_3 = \frac{f'(x_n)}{f''(x_n)}. \tag{57}$$

The errors of the above approximants are given by

$$e_n = \left| m - \widehat{m} \right| \tag{58}$$

The performance of these formulas are displayed in the table 4.

# 5. The Established Methods

For the purpose of comparison, two iterative methods presented in [10, 23] are considered. Since these methods are well established, the essential formulas are used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new twelfth order method.

The first of the method is in fact an eighth order method presented in [23] and is expressed as

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{59}$$

$$z_n = y_n - m\frac{f(y_n)}{f'(y_n)}, \tag{60}$$

$$x_{n+1} = z_n - m\frac{f(z_n)}{f'(z_n)}. \tag{61}$$

The second method is by Mir et al. and is presented in [10]. This method is actually a tenth order and is expressed as

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{62}$$

$$z_n = y_n - m\left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^{2/m}\right]\frac{f(y_n)}{f'(y_n)}, \tag{63}$$

$$x_{n+1} = z_n - m\left[1 + \left(\frac{f(z_n)}{f(y_n)}\right)^{2/m}\right]\frac{f(z_n)}{f'(z_n)}, \tag{64}$$

# 6. Numerical Results

In this section, we shall present the numerical results obtained by employing the iterative methods (7), (28), (34), (42), (61) and (64) to solve some nonlinear equations with known multiplicity $m$. To demonstrate the performance of the new higher order iterative methods, we use ten particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the higher order methods for determining the multiple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

The new twelfth order method requires six function evaluations and has the order of convergence twelve. To determine the efficiency index of the new methods, we shall use the definition 3. Hence, the efficiency index of the new methods given by (7), (28), (34), and (42) is $\sqrt[6]{12}$ whereas the efficiency index of the eighth and tenth order methods (61) and (64) is given by $\sqrt[6]{8}$ and $\sqrt[6]{10}$ respectively. We can see that the efficiency index of the new twelfth order method has better efficiency index than the eighth and tenth order method. The test functions with known multiplicities $m$ and their exact root $\alpha$ are displayed in table 1. The difference between the root $\alpha$ and the approximation $x_n$ for test functions with initial guess $x_0$ are displayed in table 2. Table 2 shows the absolute errors obtained by each of iterative methods described, we see that the new twelfth order methods are producing better results than the established methods. Furthermore, the computational order of convergence (COC) are displayed in table 3. From the table 3, we observe that the COC perfectly coincides with the theoretical result. In addition, the difference between the multiplicity $m$ and the approximation $\widehat{m}$ with initial guess $x_0$ are displayed in table 4. In table 4 we observe that there is no significant difference between the Lagouanelle formula (48) and the recently introduced formulas (52)-(56), whereas the Traub's method (51) is performing poorly. In fact, $x_n$ is calculated by using the same total number of function evaluations (TNFE) for all methods, which is after three iterations.

# 7. Conclusions

In this paper, four new twelfth order iterative methods for solving nonlinear equations with multiple roots have been introduced. Convergence analysis proves that the new methods preserve their order of convergence. Simply combining the two well-established methods, we have achieved a twelfth order of convergence. The prime motive of presenting these new methods was to establish a higher order of convergence method than the existing methods [1-28]. The effectiveness of the new twelfth order methods is examined by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that the convergence of the tested multipoint methods of the twelfth order is remarkably fast. The main purpose of demonstrating the new methods for different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative methods.

We have shown numerically, and verified, that the new iterative methods converge to the order twelfth. Empirically, we have found, in many cases that the new formulas for approximating the multiplicity *m* are performing better than the established methods. Finally, we have constructed new higher order iterative methods, but unfortunately these new methods are not of optimal order, hence further investigation is essential.

**Table 1.** Test functions, multiplicity *m*, root $\alpha$ and initial guess $x_0$

| ***Functions*** | ***m*** | ***Roots*** | ***Initial guess*** |
|---|---|---|---|
| $f_1(x)=\left(x^3+4x^2-10\right)^m$ | $m=501$ | $\alpha=1.365230...$ | $x_0=1.7$ |
| $f_2(x)=\left(xe^{x^2}-\sin(x)^2+3\cos(x)+5\right)^m$ | $m=10$ | $\alpha=-1.207647...$ | $x_0=-1.5$ |
| $f_3(x)=\left((x-1)^3-1\right)^m$ | $m=111$ | $\alpha=2$ | $x_0=2.1$ |
| $f_4(x)=\left(\exp\left(x^2+7x-30\right)-1\right)^m$ | $m=50$ | $\alpha=3$ | $x_0=3.1$ |
| $f_5(x)=\left(\cos(x)+x\right)^m$ | $m=99$ | $\alpha=-0.739085...$ | $x_0=-0.8$ |
| $f_6(x)=\left(\sin(x)^2-x^2+1\right)^m$ | $m=20$ | $\alpha=1.404491...$ | $x_0=1.8$ |
| $f_7(x)=\left(e^{-x^2}-e^{x^2}-x^8+10\right)^m$ | $m=5$ | $\alpha=1.239417...$ | $x_0=1.4$ |
| $f_8(x)=\left(6x^5+5x^4-4x^3+3x^2-2x+1\right)^m$ | $m=100$ | $\alpha=-1.57248...$ | $x_0=-1.8$ |
| $f_9(x)=\left(\tan(x)-e^x-1\right)^m$ | $m=71$ | $\alpha=1.371045...$ | $x_0=1.5$ |
| $f_{10}(x)=\left(\ln\left(x^2+3x+5\right)-2x+7\right)^m$ | $m=1000$ | $\alpha=5.469012...$ | $x_0=5.9$ |

**Table 2.** Comparison of multipoint iterative methods

| $f_i$ | (61) | (64) | (28) | (42) | (34) | (7) |
|---|---|---|---|---|---|---|
| $f_1$ | 0.186e-427 | 0.775e-759 | 0.198e-1198 | 0.248e-1241 | 0.198e-1198 | 0.318e-1284 |
| $f_2$ | 0.216e-201 | 0.120e-372 | 0.582e-642 | 0.850e-710 | 0.270e-636 | 0.196e-592 |
| $f_3$ | 0.213e-526 | 0.146e-940 | 0.251e-1534 | 0.912e-1536 | 0.265e-1534 | 0.659e-1624 |
| $f_4$ | 0.345e-148 | 0.190e-267 | 0.739e-425 | 0.292e-435 | 0.113e-424 | 0.296e-419 |
| $f_5$ | 0.227e-964 | 0.271e-1706 | 0.118e-2706 | 0.468e-2707 | 0.120e-2706 | 0.453e-2920 |
| $f_6$ | 0.612e-319 | 0.774e-561 | 0.152e-834 | 0.255e-840 | 0.405e-834 | 0.162e-904 |
| $f_7$ | 0.829e-240 | 0.194e-434 | 0.882e-724 | 0.243e-778 | 0.438e-712 | 0.298e-684 |
| $f_8$ | 0.295e-223 | 0.289e-398 | 0.501e-604 | 0.174e-606 | 0.539e-604 | 0.577e-634 |
| $f_9$ | 0.103e-86 | 0.132e-159 | 0.170e-279 | 0.317e-135 | 0.260e-279 | 0.159e-247 |
| $f_{10}$ | 0.759e-1216 | 0.331e-2095 | 0.105e-2998 | 0.105e-2998 | 0.105e-2998 | 0.137e-3484 |

**Table 3.** Performance of COC

| $f_i$ | (61) | (64) | (28) | (42) | (34) | (7) |
|---|---|---|---|---|---|---|
| $f_1$ | 8.0000 | 9.9506 | 11.960 | 11.966 | 11.960 | 12.021 |
| $f_2$ | 8.0001 | 10.357 | 12.536 | 12.426 | 12.542 | 12.667 |
| $f_3$ | 8.0000 | 9.9802 | 11.989 | 11.989 | 11.989 | 12.022 |
| $f_4$ | 8.0007 | 10.336 | 12.487 | 12.458 | 12.487 | 12.718 |
| $f_5$ | 8.0000 | 9.9905 | 11.993 | 11.993 | 11.993 | 11.996 |
| $f_6$ | 8.0000 | 9.8320 | 11.868 | 11.868 | 11.868 | 11.954 |
| $f_7$ | 8.0000 | 10.054 | 12.161 | 12.122 | 12.165 | 12.266 |
| $f_8$ | 8.0000 | 9.8930 | 11.910 | 11.909 | 11.910 | 12.089 |
| $f_9$ | 8.0149 | 13.292 | 15.398 | 24.916 | 15.402 | 16.374 |
| $f_{10}$ | 8.0000 | 9.9889 | 11.989 | 11.989 | 11.989 | 11.992 |

**Table 4.** Performance of new formulas for approximating multiplicity *m*

| $f_i$ | (56) | (49) | (55) | (53) | (54) | (52) | (51) |
|---|---|---|---|---|---|---|---|
| $f_1$ | 0.392e-49 | 0.392e-49 | 0.860e-51 | 0.860e-51 | 0.700e-51 | 0.392e-49 | 36.11 |
| $f_2$ | 0.285e-14 | 0.285e-14 | 0.389e-14 | 0.389e-14 | 0.370e-14 | 0.285e-14 | 1.36 |
| $f_3$ | 0.246e-62 | 0.246e-62 | 0.122e-62 | 0.122e-62 | 0.124e-62 | 0.246e-62 | 4.22 |
| $f_4$ | 0.507e-6 | 0.507e-6 | 0.467e-6 | 0.467e-6 | 0.469e-6 | 0.507e-6 | 13.59 |
| $f_5$ | 0.114e-118 | 0.114e-118 | 0.374e-118 | 0.374e-118 | 0.369e-118 | 0.114e-118 | 1.79 |
| $f_6$ | 0.563e-40 | 0.563e-40 | 0.186e-40 | 0.186e-40 | 0.222e-40 | 0.563e-40 | 0.81 |
| $f_7$ | 0.181e-21 | 0.181e-21 | 0.144e-21 | 0.144e-21 | 0.145e-21 | 0.181e-21 | 10.96 |
| $f_8$ | 0.140e-20 | 0.140e-20 | 0.173e-20 | 0.173e-20 | 0.173e-20 | 0.140e-20 | 14.95 |
| $f_9$ | 0.137e-10 | 0.137e-10 | 0.148e-10 | 0.148e-10 | 0.149e-10 | 0.137e-10 | 8.89 |
| $f_{10}$ | 0.121e-149 | 0.121e-149 | 0.589e-149 | 0.589e-149 | 0.588e-149 | 0.121e-149 | 21.07 |

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