

Solving Partial Integro-Differential Equations Using Double Laplace Transform Method

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Abstract In this paper, we apply the double Laplace transform for solving linear partial integro-differential equations (PIDE) with a convolution kernel. Double Laplace transform converts the PIDE to an algebraic equation which can be easily solved is illustrated by solving various examples.

Keywords Double Laplace transform, Inverse Laplace transform, Partial integro-differential equation, Partial derivatives

1. Introduction

We find in the literature, partial integro-differential equations are used in modelling various phenomena in science, engineering & social sciences. Jyoti Thorwe & Sachin Bhalekar [5] in 2012, used single Laplace transform method to solve linear partial integro-differential equations (PIDE) with a convolution kernel

$$\sum_{i=0}^m a_i \frac{\partial^i u}{\partial x^i} + \sum_{i=0}^n b_i \frac{\partial^i u}{\partial t^i} + cu + \sum_{i=0}^r d_i \int_0^t k_i(t-y) \frac{\partial^i u}{\partial x^i} dy + f(x, t) = 0, \quad (1.1)$$

(with prescribed conditions)

where $f(x, t)$ & $k_i(t - y)$ are known functions.

a_i 's, b_i 's, d_i 's & c are constants or the functions of x .

They converted the PIDE to an ordinary differential equation (ODE) using a Laplace transform. Solving this ODE & applying inverse Laplace transform they obtained the exact solution of the problem.

The double Laplace transform [4] of a function $f(x, t)$ defined in the positive quadrant of the xt -plane is defined by the equation

$$L_x L_t \{f(x, t)\} = \bar{f}(p, s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x, t) dt dx \quad (1.2)$$

whenever that integral exist. Here p, s are complex numbers.

Further the double Laplace transform [1] of first & second partial derivatives are given by

$$L_x L_t \left\{ \frac{\partial f(x, t)}{\partial x} \right\} = p \bar{f}(p, s) - \bar{f}(0, s) \quad (1.3)$$

$$L_x L_t \left\{ \frac{\partial f(x, t)}{\partial t} \right\} = s \bar{f}(p, s) - \bar{f}(p, 0) \quad (1.4)$$

$$L_x L_t \left\{ \frac{\partial^2 f(x, t)}{\partial x^2} \right\} = p^2 \bar{f}(p, s) - p \bar{f}'(0, s) - \bar{f}_x(0, s) \quad (1.5)$$

$$L_x L_t \left\{ \frac{\partial^2 f(x, t)}{\partial t^2} \right\} = s^2 \bar{f}(p, s) - s \bar{f}(p, 0) - \bar{f}_t(p, 0) \quad (1.6)$$

Double Laplace transform was used by Anwar et al. [2] to solve partial differential equations with Caputo fractional derivatives & fractional heat equation subject to certain initial & boundary conditions. Tarig M. Elazaki [8] in 2012, applied the modified variational iteration method & double Laplace transform to solve nonlinear convolution partial differential equations.

Recently in 2013, Eltayeb & Kilicman [3] have applied the double Laplace transform to solve general linear telegraph & partial integro-differential equations. Saeed Kazem [7] implemented single Laplace transform to obtain an exact solution of some linear fractional differential equations. Jafar Saberi-Nadjafi [6] proposed some new result in n -dimensional Laplace transforms involved with Fourier cosine transform; these results provide few algorithms for evaluating some n -dimensional Laplace transform pairs.

By using double Laplace transform method, we directly convert PIDE (1.1) into an algebraic equation instead of converting to ODE. Solving this algebraic equation & applying double inverse Laplace transform we obtain the exact solution. This method is illustrated by giving examples of various types already described in [5].

Convolution Theorem:

If single Laplace transform of $f(x, t)$ & $g(t)$ with respect to t are given by

$$L_t \{f(x, t)\} = \bar{f}(x, s) \text{ \& \& } L_t \{g(t)\} = \bar{g}(s) \text{ then}$$

$$L_t \{g(t) * f(x, t)\} = L_t \{g(t)\} L_t \{f(x, t)\} = \bar{g}(s) \bar{f}(x, s)$$

where, $g(t) * f(x, t) = \int_0^t g(t-y) f(x, y) dy$.

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Published online at <http://journal.sapub.org/ajcam>

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Therefore, by using double Laplace transform convolution theorem becomes

$$\begin{aligned} L_x L_t \{g(t) * f(x, t)\} &= L_x L_t \left\{ \int_0^t g(t-y) f(x, y) dy \right\} \\ &= \bar{g}(s) \bar{f}(p, s) \end{aligned} \quad (1.7)$$

2. Solving PIDEs Using Double Laplace Transform Method

Consider PIDE (1.1),

$$\begin{aligned} \sum_{i=0}^m a_i \frac{\partial^i u}{\partial x^i} + \sum_{i=0}^n b_i \frac{\partial^i u}{\partial t^i} + cu \\ + \sum_{i=0}^r d_i \int_0^t k_i(t-y) \frac{\partial^i u}{\partial x^i} dy + f(x, t) = 0, \end{aligned} \quad (2.1)$$

(with prescribed conditions)

where $f(x, t)$ & $k_i(t-y)$ are known functions & a_i, b_i, d_i, c are constants.

Taking double Laplace transform of PIDE (2.1), we get

$$\begin{aligned} \sum_{i=0}^m a_i L_x L_t \left\{ \frac{\partial^i u}{\partial x^i} \right\} + \sum_{i=0}^n b_i L_x L_t \left\{ \frac{\partial^i u}{\partial t^i} \right\} + c \bar{u}(p, s) \\ + \sum_{i=0}^r d_i L_x L_t \left\{ \int_0^t k_i(t-y) \frac{\partial^i u}{\partial x^i} dy \right\} \\ + \bar{f}(p, s) = 0 \end{aligned}$$

Using double Laplace transform of partial derivative & convolution theorem, we get

$$\begin{aligned} \sum_{i=0}^m a_i \left[p^i \bar{u}(p, s) - \sum_{j=0}^{i-1} p^{i-1-j} L_t \left\{ \frac{\partial^j}{\partial x^j} u(0, t) \right\} \right] + \\ \sum_{i=0}^n b_i \left[s^i \bar{u}(p, s) - \sum_{k=0}^{i-1} p^{i-1-k} L_x \left\{ \frac{\partial^k}{\partial t^k} u(x, 0) \right\} \right] + \\ c \bar{u}(p, s) + \\ \sum_{i=0}^r d_i \bar{k}_i(s) \left[p^i \bar{u}(p, s) - \sum_{j=0}^{i-1} p^{i-1-j} L_t \left\{ \frac{\partial^j}{\partial x^j} u(0, t) \right\} \right] + \\ \bar{f}(p, s) = 0 \end{aligned} \quad (2.2)$$

where $\bar{u}(p, s) = L_x L_t \{u(x, t)\}$, $\bar{f}(p, s) = L_x L_t \{f(x, t)\}$ & $\bar{k}_i(s) = L_t \{k_i(t)\}$

Equation (2.2) is an algebraic equation in $\bar{u}(p, s)$. Solving this algebraic equation & taking inverse double Laplace transform of $\bar{u}(p, s)$, we get an exact solution $u(x, t)$ of (2.1).

3. Illustrative Examples

Example 3.1: Consider the PIDE

$$u_{tt} = u_x + 2 \int_0^t (t-y) u(x, y) dy - 2e^x, \quad (3.1)$$

with initial condition

$$u(x, 0) = e^x, \quad u_t(x, 0) = 0 \quad (3.2)$$

& boundary condition

$$u(0, t) = \cos t. \quad (3.3)$$

Taking double Laplace transform of equation (3.1),

$$\begin{aligned} s^2 \bar{u}(p, s) - s \bar{u}(p, 0) - \bar{u}_t(p, 0) \\ = p \bar{u}(p, s) - \bar{u}(0, s) + 2 \frac{1}{s^2} \bar{u}(p, s) - \frac{2}{(p-1)s} \end{aligned} \quad (3.4)$$

And single Laplace transforms of initial conditions (3.2) & boundary condition (3.3) are given by

$$\bar{u}(p, 0) = \frac{1}{p-1}, \quad \bar{u}_t(p, 0) = 0 \quad \& \quad \bar{u}(0, s) = \frac{s}{s^2 + 1}$$

Then equation (3.4) becomes,

$$\begin{aligned} s^2 \bar{u}(p, s) - \frac{s}{p-1} \\ = p \bar{u}(p, s) - \frac{s}{s^2 + 1} + 2 \frac{1}{s^2} \bar{u}(p, s) - \frac{2}{(p-1)s} \\ \left(p + \frac{2}{s^2} - s^2 \right) \bar{u}(p, s) = \frac{s}{s^2 + 1} + \frac{2}{(p-1)s} - \frac{s}{p-1} \\ \left(\frac{ps^2 + 2 - s^4}{s^2} \right) \bar{u}(p, s) \\ = \frac{s s(p-1) + 2(s^2 + 1) - s s(s^2 + 1)}{s(p-1)(s^2 + 1)} \\ = \frac{ps^2 + 2 - s^4}{s(p-1)(s^2 + 1)} \\ \bar{u}(p, s) = \frac{1}{(p-1)} \frac{s}{(s^2 + 1)} \end{aligned} \quad (3.5)$$

Applying inverse double Laplace transform of equation (3.5), we get

$$u(x, t) = e^x \cos t \quad (3.6)$$

This is the required exact solution of equation (3.1).

Example 3.2: Consider

$$u_t - u_{xx} + u + \int_0^t e^{t-y} u(x, y) dy = (x^2 + 1)e^t - 2 \quad (3.7)$$

$$u(x, 0) = x^2, \quad u_t(x, 0) = 1, \quad (3.8)$$

$$u(0, t) = t, \quad u_x(0, t) = 0. \quad (3.9)$$

Taking double Laplace transform of equation (3.7)

$$\begin{aligned} s \bar{u}(p, s) - \bar{u}(p, 0) - \{p^2 \bar{u}(p, s) - p \bar{u}(0, s) - \bar{u}_x(0, s)\} \\ + \bar{u}(p, s) + \frac{1}{s-1} \bar{u}(p, s) = \left(\frac{2}{p^3} + \frac{1}{p} \right) \frac{1}{s-1} - \frac{2}{ps} \end{aligned} \quad (3.10)$$

And single Laplace transforms of equation (3.8) & (3.9), we get

$$\bar{u}(p, 0) = \frac{2}{p^3}, \quad \bar{u}_t(p, 0) = \frac{1}{p} \quad \& \quad \bar{u}(0, s) = \frac{1}{s^2}, \quad \bar{u}_x(0, s) = 0$$

Then equation (3.10) becomes

$$\begin{aligned} s \bar{u}(p, s) - \frac{2}{p^3} - p^2 \bar{u}(p, s) + \frac{p}{s^2} + \bar{u}(p, s) \\ + \frac{1}{s-1} \bar{u}(p, s) = \frac{(2+p^2)}{p^3(s-1)} - \frac{2}{ps} \end{aligned}$$

$$\begin{aligned} \left(s - p^2 + 1 + \frac{1}{s-1} \right) \bar{u}(p, s) \\ = \frac{(2+p^2)}{p^3(s-1)} - \frac{2}{ps} + \frac{2}{p^3} - \frac{p}{s^2} \end{aligned}$$

$$\begin{aligned}
& \left(\frac{s^2 - s - p^2 s + p^2 + s - 1 + 1}{s - 1} \right) \bar{u}(p, s) \\
&= \frac{1}{p^3 s^2 (s - 1)} \{ s^2 (p^2 + 2) - 2p^2 s (s - 1) \\
&\quad + 2s^2 (s - 1) - pp^3 (s - 1) \} \\
&\bar{u}(p, s) = \frac{-p^2 s^2 + 2p^2 s + 2s^3 - p^4 s + p^4}{p^3 s^2 (s^2 - p^2 s + p^2)} \\
&\bar{u}(p, s) = \frac{(2s^3 - 2p^2 s^2 + 2p^2 s) + (p^2 s^2 - p^4 s + p^4)}{p^3 s^2 (s^2 - p^2 s + p^2)} \\
&\bar{u}(p, s) = \frac{(2s + p^2)(s^2 - p^2 s + p^2)}{p^3 s^2 (s^2 - p^2 s + p^2)} \\
&\bar{u}(p, s) = \frac{(2s + p^2)}{p^3 s^2} = \frac{2}{p^3 s} + \frac{1}{ps^2} \quad (3.11)
\end{aligned}$$

Applying inverse double Laplace transform of equation (3.11), we get

$$u(x, t) = x^2 + t \quad (3.12)$$

This is the required exact solution of equation (3.7).

Example 3.3: Consider

$$u_t + u_{ttt} - \int_0^t \sinh(t - y) u_{xxx}(x, y) dy = 0 \quad (3.13)$$

$$u(x, 0) = 0, u_t(x, 0) = x, u_{tt}(x, 0) = 0, \quad (3.14)$$

$$u(0, t) = 0, u_x(0, t) = \sin t, u_{xx}(0, t) = 0. \quad (3.15)$$

Taking double Laplace transform of equation (3.13)

$$\begin{aligned}
& s \bar{u}(p, s) - \bar{u}(p, 0) + \{ s^3 \bar{u}(p, s) - s^2 \bar{u}(p, 0) - \\
& s \bar{u}_t(p, 0) - \bar{u}_{tt}(p, 0) \} - \frac{1}{s^2 - 1} \{ p^3 \bar{u}(p, s) - p^2 \bar{u}(0, s) - \\
& p \bar{u}_x(0, s) - \bar{u}_{xx}(0, s) \} = 0 \quad (3.16)
\end{aligned}$$

And single Laplace transforms of equation (3.14) & (3.15), we get

$$\bar{u}(p, 0) = 0, \quad \bar{u}_t(p, 0) = \frac{1}{p^2}, \quad \bar{u}_{tt}(p, 0) = 0 \text{ \& }$$

$$\bar{u}(0, s) = 0, \quad \bar{u}_x(0, s) = \frac{1}{s^2 + 1}, \quad \bar{u}_{xx}(0, s) = 0$$

Then equation (3.16) becomes

$$\begin{aligned}
& s \bar{u}(p, s) + s^3 \bar{u}(p, s) - \frac{s}{p^2} - \frac{1}{s^2 - 1} p^3 \bar{u}(p, s) \\
& + \frac{p}{(s^2 - 1)(s^2 + 1)} = 0 \\
& \left(s + s^3 - \frac{p^3}{s^2 - 1} \right) \bar{u}(p, s) = \frac{s}{p^2} - \frac{p}{(s^2 - 1)(s^2 + 1)} \\
& \left(\frac{s^3 - s + s^5 - s^3 - p^3}{s^2 - 1} \right) \bar{u}(p, s) = \frac{s^5 - s - p^3}{p^2 (s^2 - 1)(s^2 + 1)} \\
& \bar{u}(p, s) = \frac{1}{p^2 (s^2 + 1)} \quad (3.17)
\end{aligned}$$

Applying inverse double Laplace transform of equation (3.17), we get

$$u(x, t) = x \sin t. \quad (3.18)$$

This is the required exact solution of equation (3.13).

Example 3.4: Consider the PIDE

$$xu_x = u_{tt} + x \sin t + \int_0^t \sin(t - y) u(x, y) dy \quad (3.19)$$

with initial condition

$$u(x, 0) = 0, u_t(x, 0) = x \quad (3.20)$$

& boundary condition

$$u(1, t) = t. \quad (3.21)$$

Taking double Laplace transform of equation (3.19)

$$\begin{aligned}
& -\frac{\partial}{\partial p} \{ p \bar{u}(p, s) - \bar{u}(0, s) \} = s^2 \bar{u}(p, s) - s \bar{u}(p, 0) \\
& -\bar{u}_t(p, 0) + \frac{1}{p^2 (s^2 + 1)} + \frac{1}{(s^2 + 1)} \bar{u}(p, s) \quad (3.22)
\end{aligned}$$

And single Laplace transforms of equation (3.20), we get

$$\bar{u}(p, 0) = 0, \quad \bar{u}_t(p, 0) = \frac{1}{p^2}$$

Then equation (3.22) becomes

$$\begin{aligned}
& p \frac{\partial}{\partial p} \bar{u}(p, s) + \bar{u}(p, s) + s^2 \bar{u}(p, s) - \frac{1}{p^2} \\
& + \frac{1}{p^2 (s^2 + 1)} + \frac{1}{(s^2 + 1)} \bar{u}(p, s) = 0 \quad (3.23)
\end{aligned}$$

$$\frac{\partial}{\partial p} \bar{u}(p, s) + \left[\frac{s^4 + 2s^2 + 2}{s^2 + 1} \right] \frac{1}{p} \bar{u}(p, s) = \frac{1}{p^3 (s^2 + 1)}$$

This is first order linear differential equation in $\bar{u}(p, s)$.

Therefore solution of (3.23) is

$$\bar{u}(p, s) = p^{-\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]} \left\{ \int \frac{1}{p^3 (s^2 + 1)} p^{\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]} dp + C \right\}$$

where C is a constant to be determined.

$$\bar{u}(p, s) = p^{-\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]} \left\{ \frac{s^2}{(s^2 + 1)} \int p^{\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right] - 3} dp + C \right\}$$

$$\bar{u}(p, s) = \frac{s^2}{(s^2 + 1)} p^{-\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]} \frac{p^{\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right] - 2}}{\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right] - 2}$$

$$+ C p^{-\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]}$$

$$\bar{u}(p, s) = \frac{1}{p^2 s^2} + C p^{-\left[\frac{s^4 + 2s^2 + 2}{s^2 + 1}\right]} \quad (3.24)$$

From the boundary condition (3.21),

$$\bar{u}(1, s) = \frac{1}{s^2} \quad (3.25)$$

Using (3.24) & (3.25), we get $C = 0$

$$\therefore \bar{u}(p, s) = \frac{1}{p^2 s^2} \quad (3.26)$$

Applying inverse double Laplace transform of equation (3.26), we get

$$u(x, t) = xt. \quad (3.27)$$

This is the required exact solution of equation (3.19).

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