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Generalized Gaussian Elimination (GGE) Solving System of Linear Inequalities or Equalities (LIS-II)

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Abstract The author generalizes the traditional Gaussian Elimination (GE) technique to resolve the feasibility of any system of linear inequalities or equalities. Any linear system consists of either equalities and/or inequalities or mixed of both equalities and inequalities is converted to its homogeneous linear feasibility standard form (HLSF). Variable substitution (VS) in the original Gaussian Elimination is replaced by variable transition (VT) to eliminate a specific variable of choice in a recursive fashion such that at the end only one single variable left to yield the feasible interval for the selected variable if such an interval exists. Note that the feasible interval of a specific variable can be null, single value, bounded below and above, bounded below only, bounded above only, or both unbounded above and below. Furthermore, the feasible interval of a given variable if it exists must also include its integer or binary solution or solutions. It is further shown that the original GE is indeed a special case of the GGE and both GE and GGE share Identical computational complexity that is bounded by the worst case of $O\left[n^2m\right]$. GGE is applicable to any linear system with a finite number of variables, n, and m, a finite number of equalities, inequalities, or mixed constraints. GGE can be used to resolve the feasibility of a given linear system with number of variables and constraints over millions or more. The validity of GGE in dealing very large linear system not only addresses the feasibility of linear systems, it may also resolve the computational complexity of the class of NP-complete (NPC) mystery. This innovative GGE technique is applied to various linear programs with unique solution, unbounded solution, or no solution to illustrate its correctness and applicability. GGE is also shown to be applicable in resolving differential variational inequalities (DVI) for both scientific and engineering applications.

Keywords Homogenerous Linear Systems, Linear Inequalities, Feasibility, Feasible Interval, Gaussian Elimination, Differential Variation Inequalities, Linear Programs, Computational Complexity

1. Introduction

Very large system of linear inequalities with thousands or millions of variables and/or constraints are very tough to resolve. Typically, linear inequalities are solved with the famous Simplex method [1-4] the ellipsoid method [6], or the interior points method [7, 11]. Whether or not a linear system is solvable or whether or not a feasible linear system contains integer or binary solution is one of the most challenging questions known as NP-complete (NPC) for applications in operstions research (OR) [10]. In this paper, the author proposes an innovative approach that generalizes the well known Gaussian Elimination (GE) for linear equalities to resolve the feasibility problem of a system of linear inequalities. Instead of variable subsittution, it is replaced by variables transition to eliminate recursively variables of choice until only one single variable is left to determine the feasible intervl associated of the specific variable. The technique is coined as the Generalized Gaussian Elimination (GGE) for linear systems. Furthermore, it is shown that GE is simply a special case of GGE and that both GE and GGE share the same computational complexity. Simple LPs are used to illustrate for all possible cases of feasibble intervals such as single value (unique), infinite many solutions with distinct upper and lower bounds, or both bounded and unbounded cases.

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2. Homogeneous Linear Systems Feasibility (HLSF) with New Notations [17-19]

Given any system of linear inequalities or equalities in vector and matrix form,

$A\vec{x} \geq \vec{b}$ where A is an n by m matrix, $\vec{x}$ and $\vec{b}$ are m by 1 and n by 1 column vectors respectively.

We define the homogeneous linear system feasibility (HLSF) for $A\vec{x} \geq \vec{b}$ as follows

$$\vec{f} = A\vec{x} - \vec{b} = (A, -\vec{b}) * \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = L * \vec{t} \geq \vec{0} \text{ where}$$

$$L = (A, -\vec{b}) \text{ and } \vec{t} = \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \quad (1)$$

The vector $f = L * \vec{t} \geq 0$ is referred to as the feasibility vector of the original linear system of inequalities $A\vec{x} \geq \vec{b}$.

Note that any system or subsystem of linear equalities $A\vec{x} = \vec{b}$ can always be converted into a system or subsystem of linear inequalities as $f = L\vec{t} = \begin{bmatrix} A & -b \\ -A & b \end{bmatrix} \vec{t} \geq 0$.

The author adopted the following new notations to simplify the illustration of GGE:

Let $\vec{v} = (v_1, v_2, ---, v_{i-1}, v_i, v_{i+1}, ---, v_n)$ be an 1 by n row vector, and let $\overrightarrow{v_{-i}} = (v_1, v_2, ---, v_{i-1}, v_{i+1}, ----, v_n)$ be the 1 by n-1 sub-vector of $\vec{v}$.

With such a new notation, it is clear that for the dot product of two vectors $\vec{v}$ and $\vec{t}$,

we have,

$$\vec{v} \bullet \vec{t} = \sum_{k=1}^{n} v_k t_k = v_i t_i + \sum_{k \neq i}^{n} v_k t_k = v_i t_i + \overrightarrow{v_{-i}} \bullet \overrightarrow{t_{-i}} \quad (2)$$

For a linear inequality converted to dot product $\vec{v} \bullet \vec{t} \geq c$, we have $v_i t_i + \overrightarrow{v_{-i}} \bullet \overrightarrow{t_{-i}} \geq c$

Given the following homogenous linear inequality (HLI):

$$A\vec{x} - \vec{b} = (A, -\vec{b}) * \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = L * \vec{t} \geq \vec{0} \text{ where}$$

$$L = (A, -\vec{b}) \text{ and } \vec{t} = \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \quad (3)$$

$$\text{Let } L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix} = (l_{ij})_{m \times n} \text{ where}$$

$$L_j = (l_{j1}, l_{j2}, ----, l_{jn}) \quad (4)$$

We also adopt the notation, $f_{-i}(\overrightarrow{x_{-i}}) \geq 0$ to highlight the nonexistence or absence of the variable, x_i, from the function (either an equality or an inequality) for $f(\overrightarrow{x_{-i}}) \geq 0$.

2.1. A Literature Overview and Motivation

Traditionally, linear inequalities are resolved as linear programs using either the Simplex methods (Dantzig, 1947), Crisis Cross algorithms (Fukuda & Terlaky, 1997), interior point method (Khachiyan, 1979), projective algorithms (Strang, 1987), path-following algorithms (Gondzio & Terlaky, 1996), and penalty or barrier functions (Nocedal and Wright, 1999). Most approaches centered on iterative searching for feasible points within the n dimensional polytope, i.e., the n-polytope, defined by the constraint linear inequalities. The author with his colleagues [18] proposed a new approach that recursively reduces the worst infeasibility, the sum of all infeasibility, and the number of constraints with the worst infeasibility based upon nonzero coefficients or a subset of the nonzero coefficients that defines the given system of linear inequalities [18]. Such an approach is capable of finding the exact solution if such a feasible solution is unique, a feasible solution if there are more than one solution. For linear system that does not have a feasible solution, this approach is capable of minimizing the sum of all infeasibility or the worst infeasibility, and pinpointing to the relevant coefficients to reveal true conflicts of the linear system [18].

Over a five years period from 2009 to 2014 the author presented three new techniques in dealing with linear systems with both equalities and inequalities were propose by the author and his colleagues during the past few years. The first technique (LIS-I) that examines the atomic components of system of linear inequalities is a set of algorithms that recursively reduce the sum of all infeasibility, maximum infeasibility, and the number of constraints with the maximum infeasibility [17, 18], This paper details a generalized Gaussian Elimination (GGE) to obtain the feasible interval of individual variable as LIS-II. A third technique as LS-III utilizes both projective and orthogonal geometry of unit vectors over the surface an n-dimensional hyperspace as the unit-shell and the concepts of equal distanced points to selected set of points on the unit shell with increasing rank recursively to locate a solution if such a solution does exists [19].

3. The Generalized Gaussian Elimination (GGE) Algorithm for HLSF

Let a HLSF in its standard form as (1) $\vec{f} = A\vec{x} - \vec{b} = (A, -\vec{b}) * \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = L * \vec{t} \geq \vec{0}$ where the constraint matrix, Let the HLSF $f = L\vec{t} \geq 0$ with n constraints and m variables , then L, is an n by $m+1$ matrix with n linear constraints and variables with

$\vec{t}=(x_1 x_2,....,x_m,1)^T$. We have the i-th column of L, C_i, and the j-th row of L, R_j is respectively:

$$C_i=\begin{bmatrix} l_{1i} \\ l_{2i} \\ : \\ l_{(n-1)i} \\ l_{ni} \end{bmatrix} \text{ and }$$

$$R_j=\begin{bmatrix} l_{j1} & l_{j2} & \dots & l_{j(m-1)} & l_{jm} & l_{j(m+1)} \end{bmatrix}$$

To eliminate a specific variable, x_i from $f=L\vec{t}\geq 0$, GGE requires the following steps:

Step 1: normalization;

Step 2: rows permutation by sign;

Step 3: sorting of rows by decreasing number of nonzero coefficients;

Step 4: replacing rows using binding transition;

Steps 1 to 4 are repeatedly applied to eliminate other variables until only the desirable variable left.

When only one variable is left, the final HLSF uniquely defines both an upper bound and a lower bound as the feasible interval for remaining variable that is not eliminated. Note that the feasible interval $F_i=[\alpha_i,\beta_i]$ with $\beta_i \leq \alpha_i$ may have the following 6 possible cases:

(i) no solution such that $F_i=[\alpha_i,\beta_i]=\phi$ with $\beta_i<\alpha_i$

(ii) unique solution with $F_i=[\alpha_i,\beta_i]=[r,r]$, i.e., we have $x_i=r$

(iii) bounded interval with infinite many solutions as $F_i=[\alpha_i,\beta_i]$ such that $x_i\in F_i$ with $-\infty<\alpha_i<\beta_i<\infty$

(iv) bounded above as $F_i=[\alpha_i,\beta_i]$ such that $-\infty=\alpha_i<\beta_i<\infty$

(v) bounded below as $F_i=[\alpha_i,\beta_i]$ such that $-\infty<\alpha_i<\beta_i=\infty$

(vi) unbounded above and below as $F_i=[\alpha_i,\beta_i]$ such that $-\infty=\alpha_i<\beta_i=\infty$

Note: (vi) may not exist; it is listed for completeness.

Assume that a selected column variable, x_i, is to be eliminated, the detailed description of each step for the GGE is provided as follows:

Step 1: Normalization: Normalization of column C_i associated with the variable x_i is to force all nonzero coefficients of C_i has value of 1 for all positive numbers and -1 for all negative numbers and zero otherwise. It is equivalent to construct a diagonal matrix $D_i=\mathrm{Diag}(d_{11},d_{22},...,d_{(n-1)(n-1)},d_{nn})$ where $d_{kk}=1$ if $l_{ki}=0$ and $d_{kk}=1/|l_{ki}|$ if $l_{ki}\neq 0$ such that $D_i f=D_i L\vec{t}\geq 0$.

Note that Multiplying L by D_i is simply dividing the k-th row of L by 1 if $l_{ki}=0$ and the j-th row of L by $|l_{ji}|$ if $l_{ji}\neq 0$. In other words, we have:

$$D_iL=\begin{bmatrix} l_{11}/d_{11} & l_{12}/d_{11} & \dots & l_{1i}/d_{11} & \dots & l_{1(m+1)}/d_{11} \\ l_{21}/d_{22} & l_{22}/d_{22} & \dots & l_{2i}/d_{22} & \dots & l_{2(m+1)}/d_{22} \\ l_{31}/d_{33} & l_{32}/d_{33} & \dots & l_{3i}/d_{33} & \dots & l_{3(m+1)}/d_{33} \\ : & : & : & : & : & : \\ l_{(n-1)1}/d_{(n-1)(n-1)} & l_{(n-1)1}/d_{(n-1)(n-1)} & \dots & l_{(n-1)i}/d_{(n-1)(n-1)} & \dots & l_{(n-1)(m+1)}/d_{(n-1)(n-1)} \\ l_{n1}/d_{nn} & l_{n2}/d_{nn} & \dots & l_{ni}/d_{nn} & \dots & l_{n(m+1)}/d_{nn} \end{bmatrix}$$

Note that we have the i-th column of D_iL is

$$[l_{1i}/d_{11},l_{2i}/d_{22},....,l_{ii}/d_{ii},....l_{ni}/d_{nn}]^T=[s_{1i},s_{2i},...,s_{ii},...s_{ni}]^T=S_i$$

where $s_{ki}=1$ if $0<l_{ki}$, $s_{ki}=-1$ if $l_{ki}<0$, and $s_{ki}=0$ if $l_{ki}=0$, for $1\leq k\leq n$.

Upon the completion of Step 1, the i-th column, associated with the variable, x_i consists of only three values, 1, -1, or 0. Step 2 is needed to group rows in D_iL by the signs in S_i in its i-th column and also sort rows in each group by decreasing number of nonzero coefficients (nzc) of the associated row. In equation notations, We are rearranging the rows of D_iL into three parts by a row permutation matrix, P, such that

$$T = PD_iL = \begin{bmatrix} T_P \\ T_N \\ T_Z \end{bmatrix}$$ where $P = [e_{r_1}, e_{r_2}, ---, e_{r_n}]$ and the i-th column of T_P is $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

the i-th column of T_N=$\begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$, and the i-th column of T_Z=$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Furthermore, we have

Step 3. Identifying most and least binding rows: Both T_P and T_N are sorted in decreasing order of $\mathbf{nzc}(r)$ such that $r_k \le r_j$ if $\mathbf{nzc}(r_j) \ge \mathbf{nzc}(r_k)$ where $\mathbf{nzc}(r)$ is defined as the number of nonzero coefficients of the r-th row of T. We refer to rows in T_P or T_N with the maximum nzc(r) as binding rows. Two cases of binding rows must be separated properly to locate the least and most binding rows. Let row r in T_P be, we have $\sum_{k=1}^{i-1}(-p_{rk})x_k + \sum_{k=i+1}^{m}(-p_{rk})x_k - p_{r(m+1)} = f_{-i}(\overrightarrow{x_{-i}}) \le x_i$

It is possible that another binding row s in T_P as:

$$\sum_{k=1}^{i-1}p_{sk}x_k + x_i + \sum_{k=i+1}^{m}p_{sk}x_k + p_{s(m+1)} \ge 0, \text{ we have } \sum_{k=1}^{i-1}(-p_{sk})x_k + \sum_{k=i+1}^{m}(-p_{sk})x_k - p_{s(m+1)} = g_{-i}(\overrightarrow{x_{-i}}) \le x_i$$

Note that if $\sum_{k=1}^{i-1}(-p_{rk})x_k + \sum_{k=i+1}^{m}(-p_{rk})x_k = \sum_{k=1}^{i-1}(-p_{sk})x_k + \sum_{k=i+1}^{m}(-p_{sk})x_k = h_{-i}(\overrightarrow{x_{-i}})$, we have

$f_{-i}(\overrightarrow{x_{-i}}) = h_{-i}(\overrightarrow{x_{-i}}) - p_{r(m+1)} \le x_i$ and $g_{-i}(\overrightarrow{x_{-i}}) = h_{-i}(\overrightarrow{x_{-i}}) - p_{s(m+1)} \le x_i$, hence, the most lower binding (MLB) row for x_i is row j such that $p_{j(m+1)} = \max\{-p_{r(m+1)}, -p_{s(m+1)}, \dots, -p_{u(m+1)}\} = \min\{p_{r(m+1)}, p_{s(m+1)}, \dots, p_{u(m+1)}\}$.

Similarly, let row r in T_N be $\sum_{k=1}^{i-1}p_{rk}x_k - x_i + \sum_{k=i+1}^{m}p_{rk}x_k + p_{r(m+1)} \ge 0$, we have $x_i \le \sum_{k=1}^{i-1}p_{rk}x_k + \sum_{k=i+1}^{m}p_{rk}x_k + p_{r(m+1)} = f_{-i}(\overrightarrow{x_{-i}})$

It is possible that another binding row s in T_N as:

$$\sum_{k=1}^{i-1}p_{sk}x_k - x_i + \sum_{k=i+1}^{m}p_{sk}x_k + p_{s(m+1)} \ge 0, \text{ we have } x_i \le \sum_{k=1}^{i-1}p_{sk}x_k + \sum_{k=i+1}^{m}p_{sk}x_k + p_{s(m+1)} = g_{-i}(\overrightarrow{x_{-i}})$$

Note that if $\sum_{k=1}^{i-1}p_{rk}x_k + \sum_{k=i+1}^{m}p_{rk}x_k = \sum_{k=1}^{i-1}p_{sk}x_k + \sum_{k=i+1}^{m}p_{sk}x_k = \hbar_{-i}(\overrightarrow{x_{-i}})$, we have

$f_{-i}(\overrightarrow{x_{-i}}) = \hbar_{-i}(\overrightarrow{x_{-i}}) + p_{r(m+1)} \le x_i$ and $g_{-i}(\overrightarrow{x_{-i}}) = \hbar_{-i}(\overrightarrow{x_{-i}}) + p_{s(m+1)} \le x_i$, hence, the least upper binding (LUB) row for x_i is row k such that $p_{k(m+1)} = \min\{p_{r(m+1)}, p_{s(m+1)}, \dots, p_{u(m+1)}\} = \max\{-p_{r(m+1)}, -p_{s(m+1)}, \dots, -p_{u(m+1)}\}$.

Elimination of the variable x_i must preserve the most and/or least binding rows with respect to all the connected variables included and accounted for.

Step 4. Note that each row in T_P uniquely defines a lower bound of x_i with respect to all other variables in $\overrightarrow{x_{-i}} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ... x_m)$. Similarly, each row in T_N uniquely define an upper bound of x_i with respect to all other variables in $\overrightarrow{x_{-i}}$.

Select distinct binding rows from T_P and T_N such that we have:

$$\sum_{k=1}^{i-1} p_{rk}x_k + x_i + \sum_{k=i+1}^{m} p_{rk}x_k + p_{r(m+1)} \geq 0 \text{ if row } r \text{ in } T_P$$

$$\text{and } \sum_{k=1}^{i-1} p_{uk}x_k - x_i + \sum_{k=i+1}^{m} p_{uk}x_k + p_{u(m+1)} \geq 0 \text{ if row } u \text{ in } T_N$$

Consequently, we have a lower bound for x_i from a binding row in T_P as

$$\sum_{k=1}^{i-1} (-p_{rk})x_k + \sum_{k=i+1}^{m} (-p_{rk})x_k - p_{r(m+1)} = f_{-i}(\overrightarrow{x_{-i}}\} \leq x_i$$

and an upper bound for x_i from a binding row in T_N as

$$x_i \leq g_{-i}(\overrightarrow{x_{-i}}) = \sum_{k=1}^{i-1} p_{uk}x_k + \sum_{k=i+1}^{m} p_{uk}x_k + p_{u(m+1)}$$

As shown in Step 3, a unique most lower binding (MLB) row and least upper binding (LUB) row for x_i may be identified if such binding relationship exists.

From this two extreme binding rows, the variable, x_i, can be safely and successfully eliminated as:

$$f_{-i}(\overrightarrow{x_{-i}}) \leq x_i \leq g_{-i}(\overrightarrow{x_{-i}}) \Rightarrow g_{-i}(\overrightarrow{x_{-i}}) - f_{-i}(\overrightarrow{x_{-i}}) = d_{-i}(\overrightarrow{x_{-i}}) \geq 0$$

For each row α within T_P we have $\sum_{k=1}^{i-1} p_{\alpha k}x_k + x_i + \sum_{k=i+1}^{m} p_{\alpha k}x_k + p_{\alpha(m+1)} \geq 0$

i.e., $$x_i \geq v_{-i}(\overrightarrow{x_{-i}}) = \sum_{k=1}^{i-1} (-p_{\alpha k})x_k + \sum_{k=i+1}^{m} (-p_{\alpha k})x_k - p_{\alpha(m+1)}$$

Consequently, we have $f_{-i}(\overrightarrow{x_{-i}}) \geq x_i \geq v_{-i}(\overrightarrow{x_{-i}})$ i.e., $f_{-i}(\overrightarrow{x_{-i}}) - v_{-i}(\overrightarrow{x_{-i}}) = s_{-i}(\overrightarrow{x_{-i}}) \geq 0$

Similarly, for each row β within T_N we have $\sum_{k=1}^{i-1} p_{\beta k}x_k - x_i + \sum_{k=i+1}^{m} p_{\beta k}x_k + p_{\beta(m+1)} \geq 0$

i.e., $$x_i \leq w_{-i}(\overrightarrow{x_{-i}}) = \sum_{k=1}^{i-1} (p_{\beta k})x_k + \sum_{k=i+1}^{m} (p_{\beta k})x_k + p_{\beta(m+1)}$$

Consequently, we have $f_{-i}(\overrightarrow{x_{-i}}) \leq x_i \leq w_{-i}(\overrightarrow{x_{-i}})$ i.e., $w_{-i}(\overrightarrow{x_{-i}}) - f_{-i}(\overrightarrow{x_{-i}}) = t_{-x}(\overrightarrow{x_{-i}}) \geq 0$

Note that rows in T_z do not contain the variable, x_i. In other words, we have shown that the variable, x_i may be eliminated completely and safely from every row in T without loss of all the binding inequalities that include all possible lower or upper bounds in terms of the remaining variables, $\overrightarrow{x_{-i}}$

Note that steps 1 to 4 may be applied recursively to eliminate any specific ordering of variables in $\vec{x}$ such that only a specific variable, x_j left such that we have the final inequalities:

$$\begin{bmatrix} x_1 & . & x_{j-1} & x_j & x_{j+1} & . & x_{n-1} & x_n & 1 \end{bmatrix} \quad \vec{t}$$

$$\begin{bmatrix} 0 & .... & 0 & 1 & 0 & .... & 0 & 0 & p_1 \\ : & .... & 0 & : & 0 & .... & 0 & 0 & : \\ 0 & .... & 0 & 1 & 0 & .... & 0 & 0 & p_{|T_P|} \\ 0 & .... & 0 & -1 & 0 & .... & 0 & 0 & q_1 \\ 0 & .... & 0 & : & 0 & .... & 0 & 0 & : \\ 0 & .... & 0 & -1 & 0 & .... & 0 & 0 & q_{|T_N|} \\ 0 & .... & 0 & 0 & 0 & .... & 0 & 0 & r_1 \\ 0 & .... & 0 & : & 0 & .... & 0 & 0 & : \\ 0 & .... & 0 & 0 & 0 & .... & 0 & 0 & r_{|P_Z|} \end{bmatrix} \begin{bmatrix} x_1 \\ : \\ x_{j-1} \\ x_j \\ x_{j+1} \\ : \\ x_{n-1} \\ x_n \\ 1 \end{bmatrix} \geq 0 \tag{5}$$

Inequalities (5) define all possible upper bounds and lower bounds for the select variables, x_j.

In other words, we have:

$$-p_k \leq x_j \leq q_m \forall k = 1,2,...,|T_P|, m = 1,2,...,|T_N|$$

where $|T_P|$ is the number of rows in T_P and $|T_N|$ is the number of rows in T_N

Let $p^* = \max_{1 \leq k \leq |T_P|}\{-p_k\} \leq x_j \leq q^* = \min_{1 \leq m \leq |T_N|}\{q_m\}$, then $F_j = [p^*, q^*]$ uniquely defines a feasible interval for the variable, x_j. Consequently, we may have the following 6 possible cases for $F_j = [p^*, q^*]$.

(i) no solution such that $F_j = [p^*, q^*]$ if $q^* < p^*$

(ii) unique solution with $F_j = [p^*, q^*]$ with $p^* = q^* = r$ i.e., we have $x_j = r$.

(iii) bounded interval with infinite many solutions as $F_j = [p^*, q^*]$ if $p^* < q^*$ & $-\infty < p^* < q^* < \infty$.

(iv) bounded above only as $F_j = (-\infty, q^*]$ if $|T_P| = \phi$.

(v) bounded below only as $F_j = [p^*, \infty)$ if $|T_N| = \phi$.

(vi) unbounded above and below as $F_j = (-\infty, \infty)$ if $|T_P| = \phi$ and $|T_N| = \phi$.

4. Examples

To illustrate the validity, capability, and correctness of GGE, we provide feasible intervals for simple inequalities computed by GGE that are easily verifiable and sample linear programs covering with unique solution, no solution, or unbounded cases as follows:

Example #1

$$\begin{aligned} &\text{L1:} \quad x - 3y \leq 6 \\ &\text{L2:} \quad 3x + 4y \leq 12 \\ &\text{L3:} \quad 2x + y \geq 4 \end{aligned} \tag{6}$$

The homogeneous $L * t \geq 0$ is:

$$\begin{bmatrix} -1 & 3 & 6 \\ -3 & -4 & 12 \\ 2 & 1 & -4 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Eliminating x, we have the following inequalities

$$\begin{bmatrix} 0 & -5/2 & 6 \\ 0 & -11/2 & 18 \\ 0 & 7 & 8 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

This provides the feasible interval for x as $F_y = [-8/7, 12/5]$

Eliminating y, we have the following inequalities

$$\begin{bmatrix} -13/4 & 0 & 60/4 \\ -13/3 & 0 & 20 \\ 5/4 & 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

This provides the feasible interval for y as $F_x = [4/5, 60/13]$

Example #2

$$\begin{aligned} &L1: \quad 2y+3z \le 5 \\ &L2: \quad x+y+2z \le 4 \\ &L3: \quad x+2y+3z \le 7 \\ &L4: \quad x \ge 0 \\ &L5: \quad y \ge 0 \\ &L6: \quad z \ge 0 \end{aligned} \tag{10}$$

The homogeneous $L*t \ge 0$ is:

$$\begin{bmatrix} 0 & -2 & -3 & 5 \\ -1 & -1 & -2 & 4 \\ -1 & -2 & -3 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

Eliminating variables y and z with (11), we have the following inequalities:

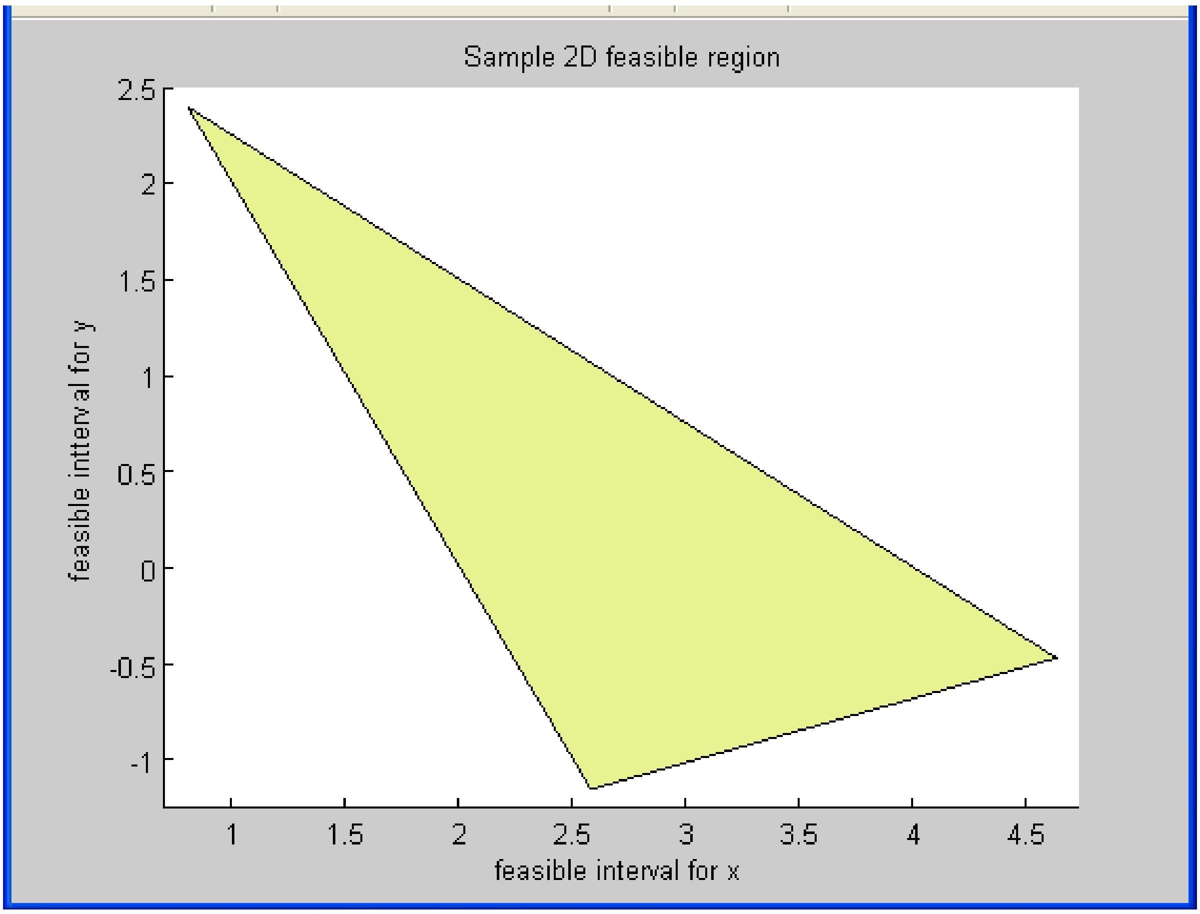

Figure 1. Feasible Intervals F_x and F_y

$$\begin{bmatrix} 0 & 0 & -3 & 5 \\ -1 & 0 & -2 & 4 \\ -1 & 0 & -3 & 7 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 5 \\ -1 & 0 & 0 & 4 \\ -1 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

This provides the feasible interval for *x* as $F_x = [0,4]$

Eliminating variable *x* and *y* with (11), we have the following inequalities:

$$\begin{bmatrix} 0 & -2 & -3 & 5 \\ 0 & -1 & -2 & 4 \\ 0 & -2 & -3 & 7 \\ 0 & -1 & -2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & -3 & 5 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -3/2 & 7/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

This provides the feasible interval for z as $F_z = [0,5/3]$

Eliminating variables x and z with (11), we have the following inequalities:

$$\begin{bmatrix} 0 & -2 & -3 & 5 \\ 0 & -1 & -2 & 4 \\ 0 & -2 & -3 & 7 \\ 0 & -1 & -2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -2 & 0 & 5 \\ 0 & -1 & 0 & 4 \\ 0 & -2 & 0 & 7 \\ 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

This provides the feasible interval for y as $F_y = [0,5/2]$

Figure 6 illustrates the feasible region for these feasible intervals.

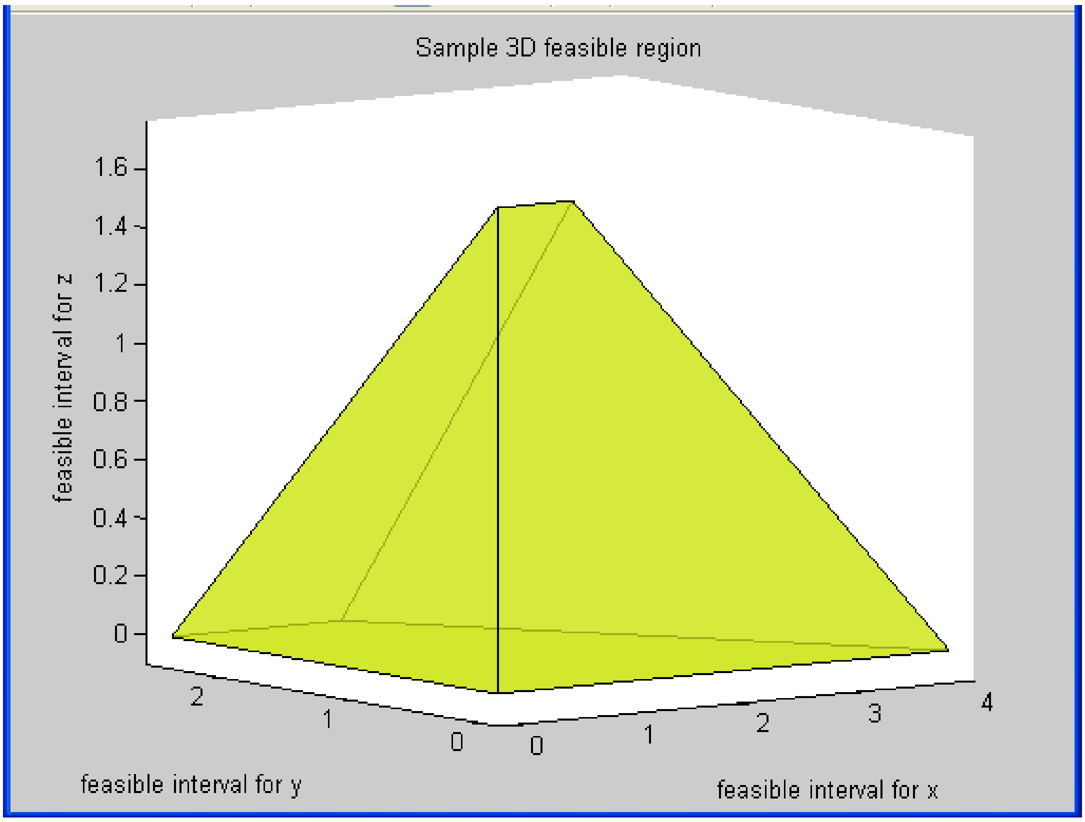

Figure 6. Feasible Intervals for F_x, F_y, and F_z

Example #3 a linear program with unique solution:

Consider the following pair of primal and dual linear programs:

$$\begin{array}{lll}
\textbf{Primary LP:} & \boldsymbol{Max} & \{c \bullet x \mid A * x \le b; l \le x \le u\} \\
\textbf{Dual LP:} & \boldsymbol{Min} & \{b * y \mid y * A = c; 0 \le y\} \\
\textbf{Optimality:} & & c * x = b * y
\end{array}$$

The self-dual LP as Homogeneous Liner Inequalities (HLI) (Wang, 2013):

$$\begin{bmatrix} 0 & -A & b \\ A^T & 0 & -c \\ -b & c & 0 \\ b & -c & 0 \\ I & 0 & 0 \\ 0 & I & -l \\ 0 & -I & u \end{bmatrix} * \begin{bmatrix} y \\ x \\ 1 \end{bmatrix} = S * t = f \ge 0$$

Consider the primal and dual LP pair:

$$\max z = -12x_1 - 18x_2 - 15x_3$$

$$\text{subject to: } \begin{bmatrix} -4 & -8 & -6 \\ -3 & -6 & -12 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} -64 \\ -96 \end{bmatrix}, \quad 0 \le \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\min w = -64y_1 - 96y_2$$

$$\text{subject to: } \begin{bmatrix} -4 & -3 \\ -8 & -6 \\ -6 & -12 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ge \begin{bmatrix} -12 \\ -18 \\ -15 \end{bmatrix}, \quad 0 \le \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The corresponding HLFS for this LP as self-dual form [18, 19] is:

$$\begin{array}{cc}
 & \begin{bmatrix} y_1 & y_2 & x_1 & x_2 & x_3 & 1 \end{bmatrix} \\
\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ . \\ r_5 \\ r_6 \\ . \\ . \\ . \\ . \\ r_{11} \\ r_{12} \end{array} &
\begin{bmatrix}
0 & 0 & 2 & 4 & 3 & -32 \\
0 & 0 & 1 & 2 & 4 & -32 \\
-4 & -3 & 0 & 0 & 0 & 12 \\
-8 & -6 & 0 & 0 & 0 & 18 \\
-6 & -12 & 0 & 0 & 0 & 15 \\
64 & 96 & -12 & -18 & -15 & 0 \\
-64 & -96 & 12 & 18 & 15 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
* \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (15)$$

Applying GGE to eliminate y_1, normalize column y_1 we have:

$$
\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ . \\ r_5 \\ r_6 \\ . \\ . \\ . \\ . \\ r_{11} \\ r_{12} \end{array}
\begin{array}{c}
\left[\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & 1 \end{array}\right] \\
\left[\begin{array}{cccccc}
0 & 0 & 2 & 4 & 3 & -32 \\
0 & 0 & 1 & 2 & 4 & -32 \\
-1 & -3/4 & 0 & 0 & 0 & 13 \\
-1 & -3/4 & 0 & 0 & 0 & 9/4 \\
-1 & -2 & 0 & 0 & 0 & 5/2 \\
1 & 3/2 & -3/16 & -9/32 & -15/64 & 0 \\
-1 & -3/2 & 3/16 & 9/32 & 15/64 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
* \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}
\geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (16)
$$

Select distinct binding rows, LUB $\boldsymbol{r_5}$ and MLB $\boldsymbol{r_6}$ to eliminate $\boldsymbol{y_1}$, we Have

$$
\begin{array}{c|cccccc}
 & y_1 & y_2 & x_1 & x_2 & x_3 & 1 \\
\end{array}
$$

$$
\begin{array}{c} r \\ r \\ r \\ : \\ r_5 \\ r_6 \\ . \\ .. \\ . \\ . \\ r_{11} \\ r_{12} \end{array}
\begin{bmatrix}
0 & 0 & 2 & 4 & 3 & -32 \\
0 & 0 & 1 & 2 & 4 & -32 \\
0 & 3/4 & -3/16 & -9/32 & -15/64 & 3 \\
0 & 3/4 & -3/16 & -9/32 & -15/64 & 9/4 \\
0 & -1/2 & -3/16 & -9/32 & -15/84 & 5/2 \\
0 & -1/2 & -3/16 & -9/32 & -15/64 & 5/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 5/2 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
* \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}
\geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (17)
$$

After normalization for of 2th column for $\boldsymbol{y_2}$, we have:

$$\begin{array}{c|cccccc|}
 & y_1 & y_2 & x_1 & x_2 & x_3 & 1 \\ \hline
r & 0 & 0 & 2 & 4 & 3 & -32 \\
r & 0 & 0 & 1 & 2 & 4 & -32 \\
r & 0 & 1 & -1/4 & -3/8 & -5/16 & 4 \\
: & 0 & 1 & -1/4 & -3/8 & -5/16 & 3 \\
r_5 & 0 & -1 & -3/8 & -9/16 & -15/32 & 5 \\
r_6 & 0 & -1 & -3/8 & -9/16 & -15/32 & 5 \\
. & 0 & 0 & 0 & 0 & 0 & 0 \\
.. & 0 & -1 & 0 & 0 & 0 & 5/4 \\
. & 0 & 1 & 0 & 0 & 0 & 0 \\
. & 0 & 0 & 1 & 0 & 0 & 0 \\
r_{11} & 0 & 0 & 0 & 1 & 0 & 0 \\
r_{12} & 0 & 0 & 0 & 0 & 1 & 0
\end{array} * \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \qquad (18)$$

Select distinct binding rows, MLB $\boldsymbol{r_4}$ and LUB $\boldsymbol{r_5}$ to eliminate $\boldsymbol{y_2}$, we have

$$\begin{array}{c|cccccc|}
 & y_1 & y_2 & x_1 & x_2 & x_3 & 1 \\ \hline
r_1 & 0 & 0 & 2 & 4 & 3 & -32 \\
r_2 & 0 & 0 & 1 & 2 & 4 & -32 \\
r_3 & 0 & 0 & -5/8 & -15/16 & -25/32 & 9 \\
r_4 & 0 & 0 & -5/8 & -15/16 & -25/32 & 8 \\
r_5 & 0 & 0 & -5/8 & -15/16 & -25/32 & 8 \\
r_6 & 0 & 0 & -5/8 & -15/16 & -25/32 & 8 \\
r_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
r_8 & 0 & 0 & -1/4 & -3/8 & -5/16 & 17/4 \\
r_9 & 0 & 0 & -3/8 & -9/16 & -15/32 & 5 \\
r_{10} & 0 & 0 & 1 & 0 & 0 & 0 \\
r_{11} & 0 & 0 & 0 & 1 & 0 & 0 \\
r_{12} & 0 & 0 & 0 & 0 & 1 & 0
\end{array} * \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \qquad (19)$$

After normalization for of 3rd column and the removal of identical rows, we have:

$$\begin{array}{c|cccccc|}
 & y_1 & y_2 & x_1 & x_2 & x_3 & 1 \\ \hline
r_1 & 0 & 0 & 1 & 2 & 3/2 & -16 \\
r_2 & 0 & 0 & 1 & 2 & 4 & -32 \\
r_3 & 0 & 0 & -1 & -3/2 & -5/4 & 72/5 \\
. & 0 & 0 & -1 & -3/2 & -5/4 & 64/5 \\
. & 0 & 0 & -1 & -3/2 & -5/4 & 17 \\
r_6 & 0 & 0 & -1 & -3/2 & -5/4 & 40/3 \\
r_7 & 0 & 0 & 1 & 0 & 0 & 0 \\
r_8 & 0 & 0 & 0 & 1 & 0 & 0 \\
r_9 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} * \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \qquad (20)$$

Select distinct binding rows, MLB $\boldsymbol{r_1}$ and LUB $\boldsymbol{r_4}$ to eliminate $\boldsymbol{x_1}$, we have

$$\begin{array}{c|cccccc|}
 & y_1 & y_2 & x_1 & x_2 & x_3 & 1 \\ \hline
r_1 & 0 & 0 & 0 & 1/2 & 1/4 & -16/5 \\
r_2 & 0 & 0 & 0 & 1/2 & 11/4 & -96/5 \\
r_3 & 0 & 0 & 0 & 1/2 & 1/4 & -8/5 \\
. & 0 & 0 & 0 & 1/2 & 1/4 & -16/5 \\
. & 0 & 0 & 0 & 1/2 & 1/4 & 1 \\
r_6 & 0 & 0 & 0 & 1/2 & 1/4 & -8/3 \\
r_7 & 0 & 0 & 0 & -3/2 & -5/4 & 64/5 \\
r_8 & 0 & 0 & 0 & 1 & 0 & 0 \\
r_9 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} * \begin{bmatrix} y_2 \\ y_1 \\ x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \qquad (21)$$

After normalization for of 4th column for x_2 and the removal of identical rows, we have:

$$\begin{array}{c} r_1\\ r_2\\ r_3\\ .\\ .\\ r_6\\ r_7\\ r_8\\ r_9 \end{array} \overset{\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & 1 \end{array}}{\begin{bmatrix} 0 & 0 & 0 & 1 & 1/2 & -32/5\\ 0 & 0 & 0 & 1 & 11/2 & -192/5\\ 0 & 0 & 0 & 1 & 1/2 & -16/5\\ 0 & 0 & 0 & 1 & 1/2 & -32/5\\ 0 & 0 & 0 & 1 & 1/2 & 2\\ 0 & 0 & 0 & 1 & 1/2 & -16/3\\ 0 & 0 & 0 & -1 & -5/6 & 128/15\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} * \begin{bmatrix} y_2\\ y_1\\ x_1\\ x_2\\ x_3\\ 1 \end{bmatrix} \geq \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} \quad (22)$$

Select distinct binding rows, MLB r_1 and LUB r_7 to eliminate x_2, we have

$$\begin{array}{c} r_1\\ r_2\\ r_3\\ r_4\\ r_5\\ r_6\\ r_7 \end{array} \overset{\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & 1 \end{array}}{\begin{bmatrix} 0 & 0 & 0 & 0 & -2/6 & 32/15\\ 0 & 0 & 0 & 0 & 7/3 & -224/15\\ 0 & 0 & 0 & 0 & -2/6 & 16/3\\ 0 & 0 & 0 & 0 & -2/6 & 98/15\\ 0 & 0 & 0 & 0 & -2/6 & 32/15\\ 0 & 0 & 0 & 0 & -5/6 & 128/15\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} * \begin{bmatrix} y_2\\ y_1\\ x_1\\ x_2\\ x_3\\ 1 \end{bmatrix} \geq \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} \quad (23)$$

After normalization for of 5th column for x_3 and the removal of identical rows, we have:

$$\begin{array}{c} r_1\\ r_2\\ r_3\\ r_4\\ r_5\\ r_6 \end{array} \overset{\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & 1 \end{array}}{\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 32/5\\ 0 & 0 & 0 & 0 & 1 & -32/5\\ 0 & 0 & 0 & 0 & -1 & 16\\ 0 & 0 & 0 & 0 & -1 & 98/15\\ 0 & 0 & 0 & 0 & -1 & 128/15\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} * \begin{bmatrix} y_2\\ y_1\\ x_1\\ x_2\\ x_3\\ 1 \end{bmatrix} \geq \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} \quad (24)$$

Hence the MLB is obtained from row r_1 and the LUB is obtained from row r_2 such that

$\mathrm{MLB} = \max\{0, 32/5\}$ and

$\mathrm{LUB} = \min\{32/5, 98/5, 128/5, 16\}$ = 6.4

Since MLB=LUB=6.4, we conclude that the variable x_3 has unique solution with $x_3 = 6.4$

From (21) with x_3 = 6.4, for x_2 the MLB is $\max\{-5.2, 2.13333, 3.2\} = 3.2$ and LUB =3.2

Hence, x_2 has unique solution x_2 = 3.2

Substituting $x_3 = 6.4$ and x_2 = 3.2 into (19), we obtain the MLB and LUB for x_1 as MLB = 0 and

LUB = $\min\{0, 0.53333, 1.5, 4.2\} = 0$; Hence, x_1 has unique solution $x_1 = 0$.

Substituting $x_3 = 6.4$, x_2 = 3.2 and x_1 = 0 into (17), we obtain the MLB and LUB for y_2 as

MLB = $\max\{0.8, 0, 0.2\} = 0.2$ and LUB = $\min\{0.2, 1.25\} = 0.2$; Hence, y_2 has the unique solution = 0.2.

Substituting, $x_3 = 6.4$ x_2 = 3.2, $x_1 = 0$, and y_2 = 0.2 into (15), we obtain the MLB and LUB for y_1 as

MLB = $\max\{0, 2.1\} = 2.1$ and LUB = $\min\{2.1, 12.85\}$ = 2.1; Hence, y_1 has unique solution $y_1 = 2.1$.

Consequently, both the primal and the dual LPs are resolved by applying the GGE algorithm to (15) with the unique optimal solution of $y_1 = 2.1,\ y_2 = 0.2,\ x_1 = 0,\ x_2 = 3.2,\ \&\ x_3 = 6.4$

Example #4 LP with no solution

Consider the LP:

$$\max\ z = x_1 + x_2 \quad \text{subject to:}$$

$$\begin{bmatrix} -1 & 2\\ 2 & 1 \end{bmatrix} * \begin{bmatrix} x_1\\ x_2 \end{bmatrix} \leq \begin{bmatrix} -6\\ 4 \end{bmatrix} \qquad x_1 \geq 0 \ \text{ and } \ x_2 \geq 0$$

The corresponding HLFS for this LP as self-dual form [19] is:

$$\begin{array}{c} r_1\\ r_2\\ r_3\\ r_4\\ r_5\\ r_6\\ r_7\\ r_8\\ r_9\\ r_{10} \end{array} \overset{\begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array}}{\begin{bmatrix} 0 & 0 & 1 & -2 & -6\\ 0 & 0 & -2 & -1 & 4\\ -1 & 2 & 0 & 0 & -1\\ 2 & 1 & 0 & 0 & -1\\ 6 & -4 & 1 & 1 & 0\\ -6 & 4 & -1 & -1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}} * \begin{bmatrix} y_1\\ y_2\\ x_1\\ x_2\\ 1 \end{bmatrix} \geq \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} \quad (25)$$

Applying the GGE algorithm to normalize the first column, we have

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \end{array} & \begin{bmatrix} 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & -2 & -1 & 4 \\ -1 & 2 & 0 & 0 & -1 \\ 1 & 1/2 & 0 & 0 & -1/2 \\ 1 & -2/3 & 1/6 & 1/6 & 0 \\ -1 & 2/3 & -1/6 & -1/6 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (26)$$

Eliminate variable y_1 by rows r_6 and r_4, we have

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{array} & \begin{bmatrix} 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & -2 & -1 & 4 \\ 0 & 5 & 0 & 0 & -3 \\ 0 & 7 & -1 & -1 & -3 \\ 0 & 1 & -1/4 & -1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (27)$$

The 2nd column for variable y_2 are all positive, hence y_2 is only bounded below, the dual LP is unbounded above. For the primal LP, it is reduced to the following linear inequalities:

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 1 & -2 & -6 \\ -2 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (28)$$

Using GGE from rows r_1 and r_2, we can eliminate x_1 to obtain the both MLB and LUB for x_2 as $0 \le x_2$ and $x_2 \le -8/5$; Consequently, the feasible interval for x_2 is $f_{x_2} = \phi$, i.e., the primal LP has no solution !

Example #5 LP with unbounded solution

Consider the LP:

max $z = 2x_1 - x_2$ **subject to:**

$$\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 1 \\ -6 \end{bmatrix} \qquad x_1 \ge 0 \ \text{ and } \ x_2 \ge 0$$

The corresponding HLFS for this LP as self-dual form [18] is:

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \end{array} & \begin{bmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & -6 \\ 1 & -2 & 0 & 0 & -2 \\ -1 & -1 & 0 & 0 & 1 \\ -1 & 6 & 2 & -1 & 0 \\ 1 & -6 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (29)$$

Using GGE from r_3 and r_5, we can eliminate y_1 and obtain

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{array} & \begin{bmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & -6 \\ 0 & 4 & 2 & -1 & -2 \\ 0 & -3 & 0 & 0 & -1 \\ 0 & 6 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (30)$$

Normalize the 2nd column for y_1, we have

$$\begin{array}{cc} & \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{array} & \begin{bmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & -6 \\ 0 & 1 & 1/2 & -1/4 & -1/2 \\ 0 & -1 & 0 & 0 & -1/3 \\ 0 & 1 & 1/3 & -1/6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (31)$$

Regardless the primal variables x_1 and x_2, the 2nd column for y_2 and the last column for MLB and LUB,

We obtain the feasible interval $f_{y_2} = \phi$ (i.e., $f = [\alpha, \beta] = \phi$ if $\beta < \alpha$) from rows r_4 and r_6 as $y_2 \leq -1/3 = \beta$ and $\alpha = 0 \leq y_2$

Eliminate y_2 by rows r_3 and r_4 and excluding the rows with dual variables y_1 and y_2 only, we have inequalities for the primal variables x_1 and x_2 as:

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & -6 \\ 1/2 & -1/4 & -5/6 \\ 1/3 & -1/6 & -1/3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (32)$$

Normalize the 3rd column for x_1, we have:

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1/2 & -3 \\ 1 & -1/2 & -5/3 \\ 1 & -1/2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (33)$$

Eliminate x_1 from rows r_1 and r_2, we have

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r \\ r \\ r_4 \end{array} & \begin{bmatrix} 0 & 3/2 & -2 \\ 0 & 1/2 & -2/3 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (34)$$

Normalize 2nd column for x_2, we have

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & 1 \end{array} \\ \begin{array}{c} r_1 \\ r \\ r \\ r_4 \end{array} & \begin{bmatrix} 0 & 1 & -4/3 \\ 0 & 1 & -4/3 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} * \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (35)$$

From (33), it is clear that x_2 is bounded below only with MLB= $\max\{-1, 0, 4/3\} = 4/3$.

Also x_2 is not bounded above with LUB = ∞. Hence, the feasible interval for x_2 is $f_{x_2} = [4/3, \infty)$.

5. GGE for Differential Varitional Inequalities (DVI)

GGE may also be applied to linear space as a normed Banach space with an inner-product operator <u,v> with norm $||u|| = \sqrt{< u, u >}$ for solving variational inequalities (VI) or differential variational inequalities (DVI) as follows [15 and 16] or Variational-like Inequalities in Banach space [25 to 29]:

Let $\phi = \{Bx \leq b\}$, where B is an $m \times n$ matrix as a nonempty convex compact polyhedron in R^n

Let F be a continuously differentiable function from ϕ into R^n with Jacobian F'.

The variational inequality problem (VIP) associated with F and ϕ is to locate a solution x^* in ϕ satisfying the variational inequality (VI): $(x^* - x)^T F(x^*) \leq 0 \forall\, x$ in ϕ. Note that in E^n, we have $||x|| = \sqrt{< x, x >} = \sqrt{x^T x}$

Let the gap function associated with a VIP be defined for x in ϕ as:

$$g(x) = \max_{y \in \phi} (x - y)^T F(x)$$

While the dual gap function associated with a VIP is defined as $\bar{\bar{g}}(x) = \max_{y \in \phi} (x - y)^T F(y)$

Using Newton's first order Taylor linear approximation around a point x_k in ϕ, a linearized VIP as LVIP can be computed iteratively for $k = 0,1,2, \ldots,$ as:

$$(x_{k+1} - y)^T (F(x_k) + F'(x_k)(x - x_k)) \leq 0 \forall y \in \phi.$$

Consider the following nonconvex, nonlinear constrained mathematical program:

$\min_{x,\lambda} h(x, \lambda) \overset{def}{=} \lambda^T (b - Bx) = x^T F(x) + b^T \lambda$ subject to $F(x) + B^T \lambda = 0$, $Bx \leq b, 0 \leq \lambda$.

Note that optimality occurs at:

$$\min_{By \leq b} y^T F(x) = \max_{\substack{B^T \mu = F(x) \\ \mu \leq 0}} b^T \mu$$

subject to: $By \leq b, B^T \mu = F(x), \& \mu \leq 0$

Consequently, we have the following homogeneous linear feasible system of inequalities $f = Lw \geq 0$

$$\text{Where } L = \begin{vmatrix} B & 0 & b \\ 0 & B^T & -F(x) \\ 0 & -B^T & F(x) \\ 0 & -I & 0 \end{vmatrix} \text{ and } w = \begin{bmatrix} y \\ \mu \\ 1 \end{bmatrix}$$

Note that $f = Lw \geq 0$ can be resolved effectively for the feasible intervals of w derived from L as described and demonstrated by the proposed GGE algorithm illustrated in

Section 3.

6. Conclusions and Future Work

In conclusion, the author proposes a generalization of the traditional Gaussian elimination (GE) for solving system of linear equalities to compute the feasible intervals of all variables to resolve the feasibility of all linear systems with both equalities and/or inequalities included. This Generalized Gaussian Elimination (GGE) for linear systems is applicable to a wide range of engineering and scientific applications and is related closely to the NPC mystery of the operations research and solvability of differential variation inequalities (DVI). Furthermore, it can be shown that GGE is indeed a special case of GE and that both GGE and GE do share the same worst case computational complexity of $O\left[n^2 m\right]$ where n is the number of variables and m is the number of constraints. This is accomplished by replacing the variable substitution of the Gaussian elimination method by variable transition such that a specific variable may be safely and recursively eliminated without losing its binding inequalities and preserving both the most lower bounds (MLB) and the least upper bound (LUB). It is shown that any system of linear system with mixed linear equalities and inequalities may be converted into its standard homogeneous form such that the proposed GGE algorithm may be applied to obtain the feasible interval of any variable of choice. From the feasible intervals of all the variables of a given linear system, one may determine whether or not it contains binary, integers, or mixed solutions. The correctness and validity of GGE is illustrated by solving sample linear programs with unique solution, unbounded solution, and no solution.

Future work of this research includes the implementation of GGE as java code and Excel VB functions for very large system of linear inequalities or mixed of linear equalities and inequalities with millions of variables and/or constraints. The author is currently verifying a parameterized GGE algorithm for solving the linear Integer programming (LIP) problem as a potential alternative or replacement for the well known branch and bound (B&B) technique.

A draft paper will be available for external review later. Funding from NSF or private foundations will be pursued to speed up the development of Java or VB functional codes for solving eigenvector systems, computing orthogonal basis, and DVI applications. GGE may also be applied to problems in operations research to reveal the availability of integer or binary solutions encountered in the NPC mystery as open issues.

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REFERENCES

[1] Strang, Gilbert, “Introduction to Applied Mathematics”, John Wiley & Sons Inc., New York, 1979.

[2] Strang, Gilbert, “Karmarkar’s algorithm and its place in applied mathematics”, The Msthematical Intelligencer 9(2): pp. 4-10, New York: Springer, 1987.

[3] Dantzig, G. G. “Maximization of a llinear function of variables subject to linear inequalities”, 1947, Published pp. 339-347, in T.C. Koopmans (ed.): Activity Analysis of Production and Allocation, Wiley & Chapman-Hall, New York-Lodon, 1951.

[4] Dantzig, G. B. “Linear Programming and Extensions”. Princeton, NJ: Princeton University Press, 1963.

[5] Fukuda, Komei and Terlaky, Tamas, “Crisis-cross methods: A fresh view on pivot algorithms”, Mathematical Programming: Series B, No. 79, Papers from 16th International Symposium on Mathematical Programming, Lausanne, 1997.

[6] Khachiyan, L. G., “Polynomial algorithms in linear programming”, U.S.S.R., Computational Mathematical and Mathematical Physics 20 (1980) pp. 53-72.

[7] Karmarkar, N., “A New Polynomial Time Algorithm for Linear Programming”, AT&T Bell Laboratories, Murray Hill, New Jersey, September, 1984.

[8] Gondzio, Jackek and Terlaky, Tamas, “A computational view of interior point method”, Advances in linear and integer programming, Oxford Lecture Series in Mathematics and its Applications, 4, New York, Osford University Press. pp. 103-144, MR1438311, 1996.

[9] Nocedal, Jorge and Wright, Stephen J,: “Numerical Optimization”, Springer Science+Business Media, Inc., 1999.

[10] Michael. R. Garey and David. S. Johnson, COMPUTERS AND INTRACTABILITY, A Guide to the Theory of NP-Completeness, Bell Laboratories, Murray Hill, New Jersey, 1979.

[11] Nemirovsky, A. and Yudin, N. “Interior-Point Polynomial Methods in Convex Programming”, Philadelphia, PA: SIAM, 1994.

[12] Alexander Schrijver, “Theory of Linear and Integer Programming”, Department of Econometrics”, Tilburg University, A Wiley-Interscience Publication, New York, 1979.

[13] Niedringhaus, W., “Stream Option Manager (SOM):

Automated Integration of Aircraft Separation, Merging, Stream Management, and Other Air Traffic Control Problems", IEEE Transactions Systems. Man & Cybernetics, Vol. 25 No. 9, Sept. 1995.

[14] Niedringhaus, W., "Maneuver Option Manager (MOM): Automated Simplification of Complex Air Traffic Control Problems", IEEE Transactions Systems. Man & Cybernetics, May 1992. Niedringhaus, W., "Maneuver Option Manager (MOM): Automated Simplification of Complex Air Traffic Control Problems", IEEE Transactions Systems. Man & Cybernetics, May 1992.

[15] Marcotte, Patrice, "A New Algorithm for Solving Variational Inequalties with Application to the Traffic Assignment Problem", Centre de Recherche sur les Transports, University de Montreal, Canada, Mathematical Programming 33 (1985) pp. 339-351, North-Holland.

[16] Sun, Min, "A New Alternating Direction Method for Co-corecive Variational Inequality Problems with Linear Equality and Inequality Constraints", pp. 161-176, Advanced Modeling and Optimization, Vol. 12, Number 2, 2010.

[17] Wang, Paul T. R., "Solving Linear Programming Problems in Self-dual Form with the Principle of Minmax", MITRE MP-89W00023, The MITRE Corporation, July, 1989.

[18] Wang, Paul T. R., Niedringhaus, William P., and McMahon, Matthew T., "A Generic Linear Inequalities Solver (LIS) with an Application for Automated Air Traffic Control". America Journal of Computational and Applied Mathematics, pp. 195-206, Volume 3, Number 4, August 2013, http://article.sapub.org/10.5923.j.ajcam.20130304.01.html

[19] Wang, Paul T. R., "Solving System of Linear Inequalities on the Surface of the Unit Shell (LIS-III)". American Journal of Computational and Applied Mathematics, pp. 97-110, Volume 3, Number 4, August 2014, http://article.sapub.org/10.5923.j.ajcam.20140403.05.html.

[20] EE236-A, Lecture 15," Self-dual formulations", University of California, Department of Electrical Engineering, 2007-08.

[21] Self-Dual Form:http://www.ee.ucla.edu/ee236a/lectures/hsd.pdf.

[22] ILOG, "Introduction to ILOG CPLEX", 2007, http://www.ilog.com/products/optimization/qa.cfm?presentation=3.

[23] ILOG, "CPLEX Barrier Optimizer", 2008, http://www.ilog.com/products/cplex/product/barrier.cfm.

[24] Steven Skiena, "LP_SOLVE: Linear Programming Code", Stony Brook University, Dept. of Computer Science", 2008 http://www.cs.sunysb.edu/~algorith/implement/lpsolve/implement.shtml.

[25] Deepmala, "A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications", Ph.D. Thesis (2014), Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India-492010.

[26] V.N. Mishra, "Some Problems on Approximations in Banach Spaces", Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee – 247 667, Uttarakhand, India.

[27] S. Husain, S. Gupta, V.N. Mishra; "An Existence Theoram of Solutions for the System of Generalized Vector Quasi-Variational-Like Inequalities", American Journal of Operations Research (AJOR), 2013, 3.329-336. DOI:10.4236/AJOR.2013.22029.

[28] S. Husain, S. Gupta, V.N. Mishra; "Generalized H(.,.)-η-Cocoercive Operators and Generalized Set-Valued Variational-Like Inclusions", Journal of Mathematics, Vol. 2013, Article3 ID 738491, 10 pages.

[29] S. Husain, S. Gupta, V.N. Mishra; "Graph Convergence for the H((.,.),(.,.))-Mixed Mapping with an Application for Solving the System of Genberalized Variational Inclusions" Fixed Point Theory and Applications 2013, 2013:304 DOI: 10.1186/1687-1812-2013-304.