

A Short Note on Solving a Class of Nonlinear Ordinary Differential Equations in Applied Physics

M. Saravi^{1,*}, M. Hermann²

¹Islamic Azad University, Nour Branch, Dept of Mathematics

²Schiller University, Fakultät für Mathematics and Informatik

Abstract In this paper we apply two procedures to find out a particular solution to a class of second-order singular nonlinear ordinary differential equations (ODEs). All illustrative examples from the available literature in applied physics show that the solutions deduced by proposed procedures are particular solutions. Overall, a successful produce of particular solutions by the proposed process itself justify the effectiveness and efficiency of the procedures.

Keywords Nonlinear ODEs, Particular solution, Ivey's equation, Thomas-Fermi's equation, Poisson-Boltzman's equation

1. Introduction

Nonlinear ordinary differential equations (ODEs) have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion and deformation. The study of nonlinear ODEs is of interest to many researchers and various methods of solution have been proposed. Surveys of the literature with numerous references, and useful bibliographies, have been given by Nayfeh [1], Sachdev [2], Jordan and Smith [3] and more recently by He [4]. The solution of nonlinear ODEs due to limitation of existing particular solutions, which play an important role in property understanding the qualitative features of many phenomena, have been one of the most time-consuming and difficult affairs among researchers of nonlinear problems. Due to such difficulties, difficult tasks appear among the researchers for nonlinear problems.

Most of the time a desire nonlinear transformations may be useful to solve a nonlinear ODE, but it may not be sufficient. In next section, beside of using a group of transformations, we use two simple procedures to get an exact solution for some equations that are important to be solved in applied physics.

2. Procedures with Applications

It will be shown that a remarkably large number of ODEs with specific physical applications is included in the following class of equations [5]:

$$y'' + \frac{a}{y}y'^2 + \frac{b}{x}y' + \frac{c}{x^2}y + dx^r y^s = 0, \quad (1)$$

where $a, b, c, r,$ and s are arbitrary constants ($r \neq -2, s \neq 1$).

One variant of this equation obtained by setting $y = e^u$ which transforms (1) to

$$u'' + (a + 1)u'^2 + \frac{b}{x}u' + \frac{c}{x^2} + dx^r e^{(s-1)u} = 0. \quad (2)$$

A large number of nonlinear ODEs available in applied physics can be demonstrated by specific choice of constants in (1) and (2). For example;

i)- *Ivey's equation*

$$y'' - \frac{y'^2}{y} + \frac{2}{x}y' + ky^s = 0,$$

Occurring in space-charge theory, is a special case of (1) with $a = -1, b = 2, c = 0, d = k, r = 0,$ and $s = 2$.

ii)- *Thomas-Fermi's equation*

$$y'' = x^{-1/2}y^{3/2},$$

with $a = b = c = 0, d = -1, r = -1/2,$ and $s = 3/2$.

iii)- *Emden-Lane-Fowler's equation*

$$y'' + \frac{2}{x}y' = y^m,$$

is a special case of (1) with $a = 0, b = 2, c = 0, d = 1, r = 0,$ and $s = m$.

iv)- *Poisson-Boltzman's equation*

$$y'' + \frac{\alpha}{x}y' = e^y,$$

where $\alpha = 0, 1, 2$ for plane, cylindrical, and spherical

* Corresponding author:

msaravi2001@yahoo.com (M. Saravi)

Published online at <http://journal.sapub.org/ajcam>

Copyright © 2014 Scientific & Academic Publishing. All Rights Reserved

symmetry, respectively.

This is a special case of (2) with $a = -1, b = \alpha, c = 0, d = -1, r = 0,$ and $s = 2$.

v)- Bratu's equation

$$y'' = -\lambda e^y,$$

This is a special case of (2) with $a = -1, b = c = 0, d = \lambda, r = 0,$ and $s = 2$.

A reliable procedure for obtaining the solution may be done by using a special ansatz given by $y = ax^n$. This idea comes from this fact that these kind of equations may be solved by choosing $y = \sum_0^\infty a_n x^n$. For first three equations if we set $y = ax^n$, then $y' = nax^{n-1}$ and $y'' = an(n-1)x^{n-2}$. Substituting in given equations and equating both sides, gives the values of a and n .

Let us consider Ivey's equation

$$y'' - \frac{y'^2}{y} + \frac{2}{x}y' + ky^s = 0.$$

Set $y = ax^n$ and substitute y, y' and y'' into equation. We obtain

$$n(n-1)ax^{n-2} - an^2x^{n-2} + 2nax^{n-2} + ka^2x^{2n} = 0.$$

That is,

$$n(n-1) - n^2 + 2n + kax^{n+2} = 0.$$

One may easily find that, $n = -2$ and $a = \frac{2}{k}$. Thus $y = \frac{2}{kx^2}$.

For last two equations, first by assuming $y = \ln u$, we transform equation in form of Eq. (2) then set $u = ax^n$ and the rest is the same as previous one. Let us consider

Poisson-Boltzman's equation with $\alpha = 2$

$$y'' + \frac{2}{x}y' = e^y.$$

If we set $y = \ln u$ we obtain, $u'' + \frac{2}{x}u' - \frac{u'^2}{u} = u^2$. Now choosing, $u = ax^n$ then $u' = anx^{n-1}$ and $u'' = an(n-1)x^{n-2}$. Substitution in equation leads to

$$n(n-1) - n^2 + 2n = ax^{n+2}.$$

It is clear that this relation can be hold if $n = a = -2$. Thus, $u = -2x^{-2}$. That is, $y = \ln\left(\frac{-2}{x^2}\right)$.

Another procedure that may be used to get a particular solution is to reduce the order of second-order ODEs by a group of transformations. The resulting first-order ODE can be written in the form

$$M(x, y)dx + N(x, y)dy = 0. \tag{3}$$

If the graphs $M(x, y) = 0$ and $N(x, y) = 0$ intersect, the intersection points could be helpful to obtain particular solutions. It should be pointed out that this precudre may be useful if such transformatoins could be done and graphs intersect. If so, this procedure can be a general method to solve Eq. (3).

We back to Poisson-Boltzman's equation and rewrite the equation as

$$x^2y'' + 2xy' = x^2e^y.$$

Suppose $u = xy'$ and $v = x^2e^y$, then the equation transforms to

$$\frac{dv}{du} = \frac{v(u+2)}{v-u}.$$

This equation could not be solved analytically, but if we write

$$(v-u)dv - v(u+2)du = 0,$$

then one may choose $v = u$ and $u = -2$, i.e., $xy' = -2$ and $x^2e^y = -2$. These two relations lead to $y = \ln\left(\frac{-2}{x^2}\right)$. This solution is the same particular solution as we got in previous one.

Let us back to Thomas-Fermi's equation

$$y'' = x^{-1/2}y^{3/2}.$$

Set $u = x^3y$ and $v = x^4y'$, then we obtain

$$\frac{dv}{du} = \frac{u^{3/2} + 4v}{v + 3u}.$$

One may write

$$\left(u^{\frac{3}{2}} + 4v\right)du - (v + 3u)dv = 0,$$

and construct

$$\begin{aligned} u^{\frac{3}{2}} + 4v &= 0 \\ v + 3u &= 0. \end{aligned}$$

It is not difficult to show that the intersection points are $(144, -532)$ and $(0, 0)$. Now if we choose $u = 144$ or $v = -532$ we obtain $y = \frac{144}{x^3}$.

3. Conclusions

In this study, the particular solutions of some nonlinear ODEs been obtained by two simple procedures. Briefly speaking, these procedures yield extended scope of applicability, simplicity, flexibility in application, and avoidance of complicated numerical and analytical integrations.

ACKNOWLEDGMENTS

The authors would like to express their thanks to Dr O. Jalili, (Islamic Azad University, Nour Branch), for his valuable comments and suggestions.

REFERENCES

- [1] Nayfeh, A.H.: Introduction to Perturbation Techniques, New York, John Wiley & Sons, (1981).
- [2] Sachdev, P. L.: Nonlinear Ordinary Differential Equations and their Applications, Marcel Dekker, Inc (1991).
- [3] Jordan, D.W., Smith, P.: Nonlinear Ordinary Differential Equations, Oxford Applied Mathematics and Computing

Science Series, 4th Edition (2007).

- [4] He, J. H.: A variational iteration method. A kind of nonlinear analytical technique Some examples, *Int. J. Nonlinear Mech.*, 34(4), 699-709, (1999).
- [5] Abdelkader, M. A.: Sequences of nonlinear differential equations with related solutions. *Ann. Mat. Pura Appl.* 81,249-259 (1969).