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# A Decomposition Algorithm for the Solution of Fractional Quadratic Riccati Differential Equations with Caputo Derivatives

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**Abstract** An Iterative Decomposition Method is applied to solve Fractional Quadratic Riccati Differential Equations in which the fractional derivatives are given in the Caputo sense. The method presents solutions as rapidly convergent infinite series of easily computable terms. Solutions obtained compared favorably with exact solutions and solutions obtained by other known methods.

**Keywords** Fractional Riccati Differential Equations, Caputo Derivatives, Iterative Decomposition Method

## 1. Introduction

Fractional Differential Equations, which are a generalization of differential equations have been applied to model many physical models very accurately. They have been used extensively in such areas as Thermal Engineering, acoustics, Electromagnetism, Control Theory of Dynamical Systems, Robotics, Viscoelasticity, Diffusion, Signal Processing, Population Dynamics and so on [4, 5, 13].

Riccati Differential Equations are named after the Italian Nobleman Jacapo Francesco Riccati (1676-1754) [4]. In [16], the fundamental theories of the Riccati equations, and diffusion processes are discussed. The one-dimensional Schrodinger equation is closely related to a Riccati Differential Equation [15]. Solitary wave solution of a nonlinear partial differential equation can be represented as a polynomial in two elementary functions satisfying a projective Riccati differential equation [7].

Riccati differential equations are known to be concerned with applications in pattern formation in dynamic games, linear systems with Markovian jumps, river flows, econometric models, stochastic systems, control theory, and diffusion problems [10, 15, 16].

Many authors and researchers have studied the analytical and approximate solutions of Riccati differential equations. Some approximate solutions have been obtained from Homotopy Perturbation method (HPM) [1, 2, 3, 12], Homotopy Analysis Method (HAM) [4] and the Variational Iteration Method (VIM) [7].

In this paper, we consider the fractional quadratic Riccati differential equation

$$D_*^{\alpha} y(x) = A(x) + B(x)y(x) + C(x)y^2(x), \quad x \in R, \quad 0 < \alpha \le 1, x > 0 \tag{1}$$

subject to the initial condition

$$y^{(k)}(0) = y_k, \quad k = 0,1,2,\cdots,n-1 \tag{2}$$

where $\alpha$ is the order of the fractional derivative, $x$ is an integer, $A(x), B(x)$ and $C(x)$ are known functions, and $y_k$ is a constant. The derivative is the Caputo-type derivatives.

It is well known that most fractional differential equations do not have analytical solutions. Several authors have considered the approximate solution (1)-(2) using modifications of various methods, which have been applied successfully to integer-order differential equations. For instance, the Fractional Variational Iteration Method (FVIM) was applied in [8]. In [11], the well-known Adomian Decomposition Method (ADM) was considered for the same problems. In [12], the Homotopy Perturbation Method (HPM) was modified to solve fractional order quadratic Riccati differential equations.

In [4], the HPM and ADM are compared for quadratic Riccati differential equations; and in [10], the Homotopy Analysis Method is applied to fractional Riccati equations. In [15], the Bernstein operational matrices are applied; motivated probably by the results in [5]. In [4], the Differential Transform Method (DTM) is applied.

In this paper, we propose to apply the Iterative

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Decomposition Method (IDM) which has been applied successfully for integer-order differential equations (see [16, 17]). The success of the method for integer-order differential equations motivated [17, 18]. The method is devoid of any form of linearization or discretization.

The rest of the paper is organized as follows:

In section 2, we give briefly definitions related to the theory of fractional calculus. In section 3, we present the solution procedure of the Iterative Decomposition Method (IDM). Numerical examples are presented in section 4 to illustrate the efficiency and accuracy of the IDM. The conclusions are then given in the in the last section 5.

## 2. Basic Definitions

In this section, we give some definitions and properties of the fractional calculus.

**Definition 2.1:**

A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in space $C_\mu^n$ if and only if $f^{(n)} \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\beta$ if $\beta \le \mu$.

**Definition 2.2:**

Let $\alpha \ge 0$, and $n < \alpha \le n+1, n \in N$. The operator ${}_aD_t^\alpha$, defined by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-x)^{n-\alpha-1} f(x)dx, \quad (3)$$

$a \le t \le b$

$$ {}_aD_t^\alpha f(t) = f(t)$$

is called the Riemann-Liouville Fractional derivative operator of order $\alpha$.

**Definition 2.3:**

The Riemann-Liouville fractional integral operator defined on $L_1[a,b]$ of order $\alpha \ge 0$ of a function $f \in C_\mu, \mu \ge -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_\circ^x (x-t)^{\alpha-1} f(t)dt, \alpha > 0, x > 0 \quad (4)$$

**Definition 2.4:**

Let $n < \alpha \le n+1, n \in N$, and $f^{(n)}(x) \in L_1[a,b]$. The operator $D_*^\alpha$ defined by

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)dx \quad (5)$$

is called the Caputo Fractional Derivative Operator of order $\alpha$.

Properties of the operator $J^\alpha$ is found in [7] and include the following

$$J^\circ f(x) = f(x) \quad (6)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \quad (7)$$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \quad (8)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \quad (9)$$

Also, if $m-1 < \alpha < m, m \in N$ and $f \in C_\mu^m, \mu \ge -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x) \quad (10)$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{n=0}^{m-1} \frac{x^n}{n!} f^{(n)}(0) \quad (11)$$

## 3. Iterative Decomposition Method

For the fractional quadratic Riccati differential equation (1), $0 < \alpha \le 1$, with initial condition $y(0) = a$, by applying the operator $J^\alpha$ to both sides of (1), we have

$$y(x) = a + J^\alpha \left\{ A(x) + B(x)y(x) + C(x)y^2(x) \right\} \quad (12)$$

The Iterative Decomposition Method [17, 18] suggests that (12) is then of the form

$$y = f + N(y) \quad (13)$$

The solution is decomposed into the infinite series of convergent terms

$$y(x) = \sum_{n=\circ}^{\infty} y_n(x) \quad (14)$$

Form (13), we have

$$N(y) = J^\alpha \left\{ A(x) + B(x)y(x) + C(x)y^2(x) \right\} \quad (15)$$

The operator $N$ is then decomposed as

$$\begin{aligned} N(y) &= N(y_\circ) + \{N(y_\circ + y_1) - N(y_\circ)\} \\ &+ \{N(y_\circ + y_1 + y_2) - N(y_\circ + y_1)\} + \cdots \end{aligned} \quad (16)$$

Let $G_\circ = N(y_\circ)$

$$G_n = N\left(\sum_{i=0}^{n}\right) - N\left(\sum_{i=0}^{n-1} y_i\right), \qquad n = 1, 2, \cdots \quad (17)$$

Then,

$$N(y) = \sum_{i=0}^{\infty} G_i$$

set $y_\circ = f$. From (13) and (14), we have

$$y_n = \frac{G_{n-1}}{\alpha_{n-1}}, \quad n = 1, 2, \cdots$$

Then,

$$y = \sum_{n=0}^{N-1} y_n \qquad (18)$$

We can then approximate the solution (14) by (18) as $N \to \infty$.

## 4. Numerical Examples

We now apply the method proposed in section 3 to solve some numerical examples to establish the accuracy and efficiency of the method.

*Example 4.1* [19]

Consider the Riccati Differential equation

$$D_*^{\alpha} y = 1 + y^2, \quad 0 < \alpha \le 1 \qquad (19)$$

subject to the initial condition $y(0) = 0$.

The exact solution for the case $\alpha = 1$ is $y(x) = \tan x$.

By the Iterative Decomposition Method (IDM),

$$\begin{aligned}
y_\circ(x) &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}, \\
y_1(x) &= \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} \\
y_2(x) &= \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)} \\
y_3(x) &= \frac{4[\Gamma(2\alpha+1)]^2\Gamma(4\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(5\alpha+1)\Gamma(9\alpha+1)} \\
&+ \frac{2[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)\Gamma(9\alpha+1)} \\
&+ \frac{2[\Gamma(2\alpha+1)]^3\Gamma(6\alpha+1)\Gamma(10\alpha+1)x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^3\Gamma(7\alpha+1)\Gamma(11\alpha+1)} \\
&+ \frac{4[\Gamma(2\alpha+1)]^2[\Gamma(4\alpha+1)]^2\Gamma(10\alpha+1)x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^2[\Gamma(5\alpha+1)]^2\Gamma(11\alpha+1)} \\
&+ \frac{2[\Gamma(2\alpha+1)]^3\Gamma(4\alpha+1)\Gamma(6\alpha+1)\Gamma(12\alpha+1)x^{13\alpha}}{[\Gamma(\alpha+1)]^7[\Gamma(3\alpha+1)]^3\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(13\alpha+1)} \\
&+ \frac{[\Gamma(2\alpha+1)]^4[\Gamma(6\alpha+1)]^2\Gamma(14\alpha+1)x^{15\alpha}}{[\Gamma(\alpha+1)]^8[\Gamma(3\alpha+1)]^4[\Gamma(7\alpha+1)]^2\Gamma(15\alpha+1)}
\end{aligned} \qquad (20)$$

Then, $y(x)$ can be approximated as

$$
\begin{aligned}
y(x) = {} & \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\
& + \frac{[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)} + \frac{4\Gamma(2\alpha+1)\Gamma(6\alpha+1)\Gamma(4\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} \\
& + \frac{4[\Gamma(2\alpha+1)]^2\Gamma(4\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(5\alpha+1)\Gamma(9\alpha+1)} + \frac{2[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)\Gamma(8\alpha+1)x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)\Gamma(9\alpha+1)} \\
& + \frac{2[\Gamma(2\alpha+1)]^3\Gamma(6\alpha+1)\Gamma(10\alpha+1)x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^3\Gamma(7\alpha+1)\Gamma(11\alpha+1)} + \frac{4[\Gamma(2\alpha+1)]^2[\Gamma(4\alpha+1)]^2\Gamma(10\alpha+1)x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^2[\Gamma(5\alpha+1)]^2\Gamma(11\alpha+1)} \\
& + \frac{2[\Gamma(2\alpha+1)]^3\Gamma(4\alpha+1)\Gamma(6\alpha+1)\Gamma(12\alpha+1)x^{13\alpha}}{[\Gamma(\alpha+1)]^7[\Gamma(3\alpha+1)]^3\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(13\alpha+1)} \\
& + \frac{[\Gamma(2\alpha+1)]^4[\Gamma(6\alpha+1)]^2\Gamma(14\alpha+1)x^{15\alpha}}{[\Gamma(\alpha+1)]^8[\Gamma(3\alpha+1)]^4[\Gamma(7\alpha+1)]^2\Gamma(15\alpha+1)} + \cdots
\end{aligned}
\tag{21}
$$

For the particular case $\alpha = 1$,

$$y(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{29x^7}{630} + \frac{157x^9}{11340} + \frac{134x^{11}}{51975} + \frac{2x^{13}}{12285} + \frac{x^{15}}{59535} + \cdots \tag{22}$$

**Table 1.** Comparison of Solutions of Example 1 for $\alpha = 1$

| x | Exact | Approx. Soln By IDM | Error |
|---|---|---|---|
| .0 | 0.0000000000 | 0.0000000000 | 0.0000000 |
| .1 | 0.1003346721 | 0.1003346713 | 8.162E-10 |
| .2 | 0.2027100355 | 0.2027099297 | 1.0580E-7 |
| .3 | 0.3093362496 | 0.3093343442 | 1.9050E-6 |
| .4 | 0.4227932187 | 0.4227777155 | 1.5500E-5 |
| .5 | 0.5463024898 | 0.5462212762 | 8.1210E-5 |
| .6 | 0.6841368083 | 0.6838056920 | 3.3110E-4 |
| .7 | 0.8422883805 | 0.8411449022 | 1.1430E-3 |
| .8 | 1.0296385570 | 1.0261001110 | 3.5380E-3 |
| .9 | 1.2601582180 | 1.2499664940 | 1.1090E-3 |
| 1.0 | 1.5574077250 | 1.5293009690 | 2.8110E-3 |

**Example 4.2:** [9]

Consider the fractional Riccati Differential Equation

$$D_*^{\alpha} = -y^2(x) + 1, \quad 0 < \alpha \le 1 \tag{23}$$

with initial condition $y(0) = 0$.

The exact solution for the case $\alpha = 1$ is $y(x) = \dfrac{e^{2t} - 1}{e^{2t} + 1}$.

Applying the inverse operator to both sides of (23),

$$y_\circ = \frac{x^\alpha}{\Gamma(\alpha+1)}$$

$$y_1(x) = -\frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)}$$

$$y_2(x) = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} - \frac{[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)}$$

$$\begin{aligned} y_3(x) = &-\frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} \\ &+\frac{2[\Gamma(2\alpha+1)]^2\Gamma(8\alpha+1)\left\{\Gamma(5\alpha+1)\Gamma(6\alpha+1)+2\Gamma(4\alpha+1)\Gamma(7\alpha+1)\right\}x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(9\alpha+1)} \\ &-\frac{[\Gamma(2\alpha+1)]^2[\Gamma(10\alpha+1)]\left\{\Gamma(2\alpha+1)[\Gamma(5\alpha+1)]^2\Gamma(6\alpha+1)+4[\Gamma(4\alpha+1)]^2\Gamma(7\alpha+1)\right\}x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^3[\Gamma(5\alpha+1)]^2\Gamma(7\alpha+1)\Gamma(11\alpha+1)} \\ &+\frac{4[\Gamma(2\alpha+1)]^3\Gamma(4\alpha+1)\Gamma(6\alpha+1)\Gamma(12\alpha+1)x^{13\alpha}}{[\Gamma(\alpha+1)]^7[\Gamma(3\alpha+1)]^3\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(13\alpha+1)} \\ &-\frac{[\Gamma(2\alpha+1)]^4[\Gamma(6\alpha+1)]^2\Gamma(14\alpha+1)x^{15\alpha}}{[\Gamma(\alpha+1)]^8[\Gamma(3\alpha+1)]^4[\Gamma(7\alpha+1)]^2\Gamma(15\alpha+1)} \end{aligned} \tag{24}$$

Then $y(x)$ can be approximated as

$$\begin{aligned} y(x) = &\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)x^{5\alpha}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \\ &-\frac{\Gamma(2\alpha+1)\Gamma(6\alpha+1)\Gamma(2\alpha+1)\Gamma(5\alpha+1)+4\Gamma(3\alpha+1)\Gamma(4\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} \\ &+\frac{2[\Gamma(2\alpha+1)]^2\Gamma(8\alpha+1)\left\{\Gamma(5\alpha+1)\Gamma(6\alpha+1)+2\Gamma(4\alpha+1)\Gamma(7\alpha+1)\right\}x^{9\alpha}}{[\Gamma(\alpha+1)]^5[\Gamma(3\alpha+1)]^2\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(9\alpha+1)} \\ &-\frac{[\Gamma(2\alpha+1)]^2[\Gamma(10\alpha+1)]\left\{\Gamma(2\alpha+1)[\Gamma(5\alpha+1)]^2\Gamma(6\alpha+1)+4[\Gamma(4\alpha+1)]^2\Gamma(7\alpha+1)\right\}x^{11\alpha}}{[\Gamma(\alpha+1)]^6[\Gamma(3\alpha+1)]^3[\Gamma(5\alpha+1)]^2\Gamma(7\alpha+1)\Gamma(11\alpha+1)} \\ &+\frac{4[\Gamma(2\alpha+1)]^3\Gamma(4\alpha+1)\Gamma(6\alpha+1)\Gamma(12\alpha+1)x^{13\alpha}}{[\Gamma(\alpha+1)]^7[\Gamma(3\alpha+1)]^3\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(13\alpha+1)} \\ &-\frac{[\Gamma(2\alpha+1)]^4[\Gamma(6\alpha+1)]^2\Gamma(14\alpha+1)x^{15\alpha}}{[\Gamma(\alpha+1)]^8[\Gamma(3\alpha+1)]^4[\Gamma(7\alpha+1)]^2\Gamma(15\alpha+1)} + \cdots \end{aligned} \tag{25}$$

**Table 2.** Approximate Solutions for Example 4.2 for the case $\alpha = 0.5$ *and* $1$

| x | Exact α=1 | Approx. by IDM α=1 | Approx.VIM [8] α=1 | Approx. by IDM α=0.5 | Approx by VIM [8] α=0.5 |
|---|---|---|---|---|---|
| .0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| .1 | 0.099667 | 0.099668 | 0.099667 | 0.083613 | 0.086513 |
| .2 | 0.197375 | 0.197373 | 0.197375 | 0.152754 | 0.161584 |
| .3 | 0.291312 | 0.291295 | 0.291320 | 0.217856 | 0.238256 |
| .4 | 0.379948 | 0.379933 | 0.380005 | 0.300562 | 0.321523 |
| .5 | 0.462117 | 0.462417 | 0.462375 | 0.400376 | 0.413682 |
| .6 | 0.537049 | 0.536867 | 0.537923 | 0.509746 | 0.515445 |
| .7 | 0.604367 | 0.606782 | 0.606768 | 0.619649 | 0.626403 |
| .8 | 0.664036 | 0.667654 | 0.669695 | 0.737509 | 0.745278 |
| .9 | 0.716297 | 0.712766 | 0.728139 | 0.859919 | 0.870074 |
| 1.0 | 0.761594 | 0.774428 | 0.784126 | 0.979442 | 0.998176 |

For the particular case $\alpha = 1$,

$$y(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{63} + \frac{38x^9}{2835} - \frac{341x^{11}}{1134000} + \frac{4x^{13}}{12285} - \frac{x^{15}}{59535} + \cdots \tag{26}$$

**Example 4.3**

Consider the Fractional Quadratic Riccati Differential Equation [5].

$$D^{\alpha} y(x) = 2y(x) - y^2(x) + 1, \quad o < \alpha \le 1 \tag{27}$$

with initial condition $y(0) = 0$

The exact solution for the case $\alpha = 1$ is

$$y(x) = 1 + \sqrt{2}\tanh\left\{\sqrt{2}x + \frac{1}{2}\log\left[\frac{(\sqrt{2}-1)}{\sqrt{2}+1}\right]\right\}$$

Applying the inverse operator to both sides of (27) and using the initial condition given

$$y(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + J^{\alpha}\left\{2y(x) - y^2(x)\right\}$$

Taking $y_{\circ}(x) = \dfrac{x^{\alpha}}{\Gamma(\alpha+1)}$, by the IDM, we have

$$\begin{aligned} y_1(x) &= \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} \\ y_2(x) &= \frac{4x^{3\alpha}}{\Gamma(3\alpha+1)} - \left\{\frac{2[\Gamma(2\alpha+1)]^2 + 4\Gamma(\alpha+1)\Gamma(3\alpha+1)}{[\Gamma(\alpha+1)]^2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right\}x^{4\alpha} \\ &+ \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)\{1-2\Gamma(3\alpha+1)\}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha} \\ &+ \frac{4\Gamma(5\alpha+1)x^{6\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)\Gamma(6\alpha+1)} - \frac{[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)} \\ &\vdots \end{aligned} \tag{28}$$

Then, $y(x)$ can be approximated as

$$y(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \frac{4x^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$-\left\{\frac{2[\Gamma(2\alpha+1)]^2 + 4\Gamma(\alpha+1)\Gamma(3\alpha+1)}{[\Gamma(\alpha+1)]^2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right\}x^{4\alpha} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)\{1-2\Gamma(3\alpha+1)\}}{[\Gamma(\alpha+1)]^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha}$$
$$+\frac{4\Gamma(5\alpha+1)x^{6\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)\Gamma(6\alpha+1)} - \frac{[\Gamma(2\alpha+1)]^2\Gamma(6\alpha+1)x^{7\alpha}}{[\Gamma(\alpha+1)]^4[\Gamma(3\alpha+1)]^2\Gamma(7\alpha+1)}$$

For the particular case $\alpha = 1$, we have

$$y(x) = x + x^2 + \frac{x^3}{3} - \frac{2x^4}{3} - \frac{22}{15}x^5 + \frac{x^6}{9} - \frac{x^7}{63}$$

**Table 3.** Approximate Solutions of Example 4.3 for values of α

| X | y(x) Exact α=1 | y(x) Approx.IDM α=1 | y(x) IDM α=0.5 | y(x) α=0.5 [10] | y(x) IDM α=0.75 | y(x) α=0.75 [10] |
|---|---|---|---|---|---|---|
| 0.0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.110295 | 0.110267 | 0.494672 | 0.577431 | 0.233598 | 0.244460 |
| 0.2 | 0.241976 | 0.241778 | 0.904529 | 0.912654 | 0.447051 | 0.469709 |
| 0.3 | 0.395104 | 0.395002 | 1.154389 | 1.166253 | 0.654329 | 0.698718 |
| 0.4 | 0.567812 | 0.566839 | 1.353673 | 1.353549 | 0.899065 | 0.924319 |
| 0.5 | 0.756014 | 0.755839 | 1.483765 | 1.482633 | 1.134787 | 1.137952 |
| 0.6 | 0.953566 | 0.953409 | 1.558548 | 1.559656 | 1.330745 | 1.331462 |
| 0.7 | 1.152946 | 1.152645 | 1.579990 | 1.589984 | 1.470879 | 1.497600 |
| 0.8 | 1.346363 | 1.344702 | 1.600439 | 1.578559 | 1.619521 | 1.630234 |
| 0.9 | 1.526911 | 1.524688 | 1.671108 | 1.530028 | 1.699098 | 1.724439 |
| 1.0 | 1.689498 | 1.683289 | 1.779976 | 1.448805 | 1.769830 | 1.776542 |

**Example 4.4**

Consider the Fractional Quadratic Riccati Differential Equation [5]

$$D^{\alpha} y(x) = x^2 + y^2(x), \quad o < \alpha \le 1 \tag{29}$$

with initial condition $y(0) = 1$

The exact solution for the case $\alpha = 1$ is

$$y(x) = \frac{t[J_{-\frac{3}{4}}(\frac{t^2}{2})\Gamma(\frac{1}{4}) + 2J_{\frac{3}{4}}]\left(\frac{t^2}{2}\right)\Gamma\left(\frac{3}{4}\right)}{t[J_{\frac{1}{4}}(\frac{t^2}{2})\Gamma(\frac{1}{4}) - 2J_{-\frac{1}{4}}]\left(\frac{t^2}{2}\right)\Gamma\left(\frac{3}{4}\right)}$$

Applying the inverse operator to both sides of (29),

$$y(x) = 1 + \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} + J^{\alpha}\left\{y^2(x)\right\} \tag{30}$$

Taking

$$y_{\circ}(x) = 1 + \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)},$$

we have

$$y_1(x) = J^{\alpha}\left\{y_{\circ}^{2}(x)\right\}$$

$$= \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{4x^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{4\Gamma(2\alpha+5)x^{3\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+3)} \tag{31}$$

$$\begin{aligned} y_2(x) = {} & \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \frac{8x^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+3)x^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\ & + \frac{8\Gamma(3\alpha+3)x^{4\alpha+2}}{\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(4\alpha+3)} + \frac{16\Gamma(3\alpha+5)x^{4\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+3)\Gamma(4\alpha+5)} \\ & + \frac{8\Gamma(2\alpha+5)x^{4\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(4\alpha+5)} + \frac{16\Gamma(4\alpha+5)x^{5\alpha+4}}{[\Gamma(2\alpha+3)]^2\Gamma(5\alpha+5)} \\ & + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+5)x^{5\alpha+4}}{\Gamma(\alpha+1)[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)\Gamma(5\alpha+5)} + \frac{16\Gamma(2\alpha+5)\Gamma(4\alpha+7)x^{5\alpha+6}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)\Gamma(5\alpha+7)} \\ & + \frac{32\Gamma(2\alpha+5)\Gamma(5\alpha+7)x^{6\alpha+6}}{[\Gamma(\alpha+3)]^2\Gamma(2\alpha+3)\Gamma(3\alpha+5)\Gamma(6\alpha+7)} + \frac{16[\Gamma(2\alpha+5)]^2\Gamma(6\alpha+9)x^{7\alpha+8}}{[\Gamma(\alpha+3)]^4[\Gamma(3\alpha+5)]^2\Gamma(7\alpha+9)} \end{aligned} \tag{32}$$

Then, by the IDM, $y(x)$ can be approximated as

$$\begin{aligned} y(x) = {} & 1 + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4x^{2\alpha+2}}{\Gamma(2\alpha+3)} \\ & + \frac{\Gamma(2\alpha+1)x^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} + \frac{8x^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+3)x^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\ & + \frac{4\Gamma(2\alpha+5)x^{3\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+3)} + \frac{8\Gamma(3\alpha+3)x^{4\alpha+2}}{\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(4\alpha+3)} \\ & + \frac{16\Gamma(3\alpha+5)x^{4\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+3)\Gamma(4\alpha+5)} + \frac{8\Gamma(2\alpha+5)x^{4\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(4\alpha+5)} \\ & + \frac{16\Gamma(4\alpha+5)x^{5\alpha+4}}{[\Gamma(2\alpha+3)]^2\Gamma(5\alpha+5)} + + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+5)x^{5\alpha+4}}{\Gamma(\alpha+1)[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)\Gamma(5\alpha+5)} \\ & + \frac{16\Gamma(2\alpha+5)\Gamma(4\alpha+7)x^{5\alpha+6}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)\Gamma(5\alpha+7)} + \frac{32\Gamma(2\alpha+5)\Gamma(5\alpha+7)x^{6\alpha+6}}{[\Gamma(\alpha+3)]^2\Gamma(2\alpha+3)\Gamma(3\alpha+5)\Gamma(6\alpha+7)} \\ & + \frac{16[\Gamma(2\alpha+5)]^2\Gamma(6\alpha+9)x^{7\alpha+8}}{[\Gamma(\alpha+3)]^4[\Gamma(3\alpha+5)]^2\Gamma(7\alpha+9)} \end{aligned}$$

For the particular case $\alpha = 1$,

$$y(x) = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{6} + \frac{x^5}{5} + \frac{x^6}{18} + \frac{x^7}{63} + \frac{x^8}{56} + \frac{5x^9}{756} + \frac{2x^{11}}{2079} + \frac{x^{12}}{2268} + \frac{x^{15}}{59535}$$

| x | Exact | Approx. Solution By IDM |
|---|---|---|
| .1 | 0.110295 | 0.110265 |
| .2 | 0.241976 | 0.241564 |
| .3 | 0.395104 | 0.393354 |
| .4 | 0.567812 | 0.563330 |
| .5 | 0.756014 | 0.747445 |
| .6 | 0.953566 | 0.939972 |
| .7 | 1.152946 | 1.133622 |
| .8 | 1.346363 | 1.319708 |
| .9 | 1.526911 | 1.488325 |
| 1.0 | 1.689498 | 1.628571 |

## 5. Conclusions

In this paper, the Iteration Decomposition Method has been successfully applied to find approximate solutions of fractional quadratic Riccati Differential Equations.

The IDM is effective for Riccati differential equations, and hold very great promise for its applicability to other nonlinear fractional differential equations. The four examples used indicate the efficiency and accuracy of the method for fractional quadratic Riccati differential equations. The results obtained are in very acceptable agreement with those obtained by other known methods.

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