

New Ninth Order Iterative Methods for Finding Multiple Roots of Nonlinear Equations

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Abstract In this paper, four new ninth order iterative methods for solving nonlinear equations are constructed. It is proved that these methods have the convergence order of nine requiring six function evaluations per iteration. Numerical comparisons are included to demonstrate exceptional convergence speed of the proposed methods.

Keywords Modified Newton method, Root-finding, Nonlinear equations, Multiple roots, Order of convergence, Efficiency index

1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a multiple root α of multiplicity m , i.e., $f^{(j)}(\alpha) = 0$, $j = 0, 1, \dots, m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation

$$f(x) = 0, \quad (1)$$

where $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function on an open interval I and it is sufficiently smooth in the neighbourhood of α . In recent years, some modifications of the Newton method for multiple roots have been proposed and analysed [1-21]. However, there are not many methods known to handle the case of multiple roots, hence we present higher order methods for finding multiple zeros of a nonlinear equation, based on six evaluations of the function per iteration. In addition, the new ninth order methods have a better efficiency index than the eighth order methods presented in this paper.

We begin with the well-known modified Newton's method for finding multiple roots, which is given by

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (2)$$

which converges quadratically [6, 19]. Furthermore, in [21] Wu et al. introduced a fourth order Newton method for computing multiple roots, which is given as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (3)$$

$$x_{n+1} = y_n - m \frac{f(y_n)}{f'(y_n)}, \quad (4)$$

x_0 is the initial guess, provided that the denominators of (4) are not equal to zero. Simply repeating the scheme (2), the fourth order convergence method is formed. For the purpose of this paper, we repeat once more to obtain an eighth-order method. Hence we have constructed eighth-order methods for finding multiple roots of nonlinear equations. In order to construct the new eighth order method for finding multiple roots we use the concept applied in [21], therefore the new eighth order method is expressed as

$$\phi(x) = y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (5)$$

$$z_n = y_n - m \frac{f(y_n)}{f'(y_n)}, \quad (6)$$

$$x_{n+1} = z_n - m \frac{f(z_n)}{f'(z_n)}, \quad (7)$$

x_0 is the initial guess, provided that the denominators of (5) - (7) are not equal to zero.

Theorem 1

Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I . If x_0 is sufficiently close to α , suppose that the order of convergence of the iterative method $x_{n+1} = \phi(x_n)$ is k , then the convergence rate of

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Published online at <http://journal.sapub.org/ajcam>

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$x_{n+1} = \phi(\phi(\phi(x_n)))$ is k^3 .

Proof

Since the convergence rate of $x_{n+1} = \phi(x_n)$ is k , and the error equation of $x_{n+1} = \phi(\phi(x_n))$ can be written as

$$e_{n+1} = \lambda_1 \widehat{e}_n^k, \quad (8)$$

where

$$\widehat{e}_n = \lambda_2 e_n^k, \quad (9)$$

Substituting (9) into (8), the error equation of $x_{n+1} = \phi(\phi(x_n))$ is

$$e_{n+1} = \lambda_3 e_n^{k^2}, \quad (10)$$

Repeating the process once more, the error equation of $x_{n+1} = \phi(\phi(\phi(x_n)))$ is

$$e_{n+1} = \lambda_4 e_n^{k^3}, \quad (11)$$

where $\lambda_1, \lambda_2, \lambda_3 \dots$ are constants.

The error equation (11) establishes the k^3 order convergence of the new method.

Theorem 2

Assume that the function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. If $f(x)$ is sufficiently smooth in the neighbourhood of the root α , then the method defined by (7) is of order eight.

Proof

It is well established that the order of convergence of the modified Newton method (5) is 2, thus $k = 2$. Therefore, from theorem 1, the iterative method defined by (7) has eighth order convergence.

The prime motive of this study is to develop a new class of multi-step methods for finding multiple roots of nonlinear equations. In order to construct the new ninth order method for finding multiple roots we use the well-established third order methods given [2, 8, 11, 19]. The ninth and eighth order methods presented in this paper are based on six evaluations of the function per iteration. Consequently, the new ninth order methods have a better efficiency index than the eighth order method.

The paper is organized as follows. In section 2 we state some of the definitions applicable to the methods proposed in this paper. In section 3, four new iterative methods of ninth order are constructed simply by transforming the well-established third order methods furthermore convergence analysis is provided to establish the ninth order convergence. Finally, in section 4, several numerical examples are given to demonstrate the performance of the

new ninth order methods.

2. Basic Definitions

In order to establish the order of convergence of the new ninth order methods, we state some of the definitions:

Definition 1 Let $f(x)$ be a real-valued function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \quad (12)$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant [6, 19].

Definition 2 Let $e_k = x_k - \alpha$ be the error in the k th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \quad (13)$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method [6, 19].

Definition 3 Let r be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index and defined as

$$\sqrt[r]{p}, \quad (14)$$

where p is the order of the method [6, 19].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1), then the computational order of convergence [20] may be approximated by

$$\text{COC} \approx \frac{\ln \left| (x_{n+1} - \alpha)(x_n - \alpha)^{-1} \right|}{\ln \left| (x_n - \alpha)(x_{n-1} - \alpha)^{-1} \right|}, n \in \mathbb{N}. \quad (15)$$

3. Development of the Methods and Convergence Analysis

In this section we define new ninth order methods for finding multiple roots of a nonlinear equation. The new ninth order methods are based on well-known third order methods [2, 8, 11, 19], namely the Osada third order method, the Halley third order method, the Euler-Chebyshev third order method, the Chun-Neta third order method. We use these methods to construct the ninth order methods.

3.1. The Osada Ninth Order Method

The first of the new ninth order methods is based on the Osada third order method [11]. The new ninth order iterative method is given as,

$$\phi(x) = y_n = x_n - \left(\frac{m}{2}\right)(m+1)\left(\frac{f(x_n)}{f'(x_n)}\right) + \left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(x_n)}{f''(x_n)}\right), \tag{16}$$

$$x_{n+1} = y_n - \left(\frac{m}{2}\right)(m+1)\left(\frac{f(y_n)}{f'(y_n)}\right) + \left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(y_n)}{f''(y_n)}\right), \tag{17}$$

x_0 is the initial guess, provided that the denominators of (17) are not equal to zero.

3.2. The Halley Ninth Order Method

The second of the new ninth order methods is based on the Halley third order method [8]. The new ninth order iterative method is given as,

$$\phi(x) = y_n = x_n - \left(\frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}\right), \tag{18}$$

$$x_{n+1} = y_n - \left(\frac{f(y_n)}{\frac{m+1}{2m}f'(y_n) - \frac{f(y_n)f''(y_n)}{2f'(y_n)}}\right), \tag{19}$$

x_0 is the initial guess, provided that the denominators of (19) are not equal to zero.

3.3. The Euler-Chebyshev Ninth Order Method

The third of the new ninth order methods is based on the Euler-Chebyshev third order method [19]. The new ninth order iterative method is given as,

$$\phi(x) = y_n = x_n - \left(\frac{m}{2}\right)(3-m)\left(\frac{f(x_n)}{f'(x_n)}\right) - \left(\frac{m^2}{2}\right)\left(\frac{f(x_n)f''(x_n)}{f'(x_n)^3}\right), \tag{20}$$

$$x_{n+1} = y_n - \left(\frac{m}{2}\right)(3-m)\left(\frac{f(y_n)}{f'(y_n)}\right) - \left(\frac{m^2}{2}\right)\left(\frac{f(y_n)f''(y_n)}{f'(y_n)^3}\right), \tag{21}$$

x_0 is the initial guess, provided that the denominators of (21) are not equal to zero.

3.4. The Chun-Neta Ninth Order Method

The fourth of the new ninth order methods is based on the Chun-Neta third order method [2]. The new ninth order iterative method is given as,

$$\phi(x) = y_n = x_n - \left(2m^2 f(x_n)^2 f''(x_n)\right) \left[m(3-m) \times f(x_n) f'(x_n) f''(x_n) + (m-1)^2 f'(x_n)^3 \right]^{-1} \tag{22}$$

$$x_{n+1} = y_n - \left(2m^2 f(y_n)^2 f''(y_n)\right) \left[m(3-m) f(y_n) \times f'(y_n) f''(y_n) + (m-1)^2 f'(y_n)^3 \right]^{-1}, \tag{23}$$

x_0 is the initial guess, provided that the denominators of (23) are not equal to zero. The above formula is a ninth order convergence method and is efficient in obtaining a multiple root of a nonlinear equation.

In numerical mathematics it is essential to know the behaviour of an approximate method. Hence, we prove the order of convergence of the new ninth order methods. The following theorem applies to the ninth order methods formed in this paper.

Theorem 3

Assume that the function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. If $f(x)$ is sufficiently smooth in the neighbourhood of the root α , then the methods defined by (17), (19), (21), and (23) is of order nine.

Proof

It is well established that the order of convergence of the methods (16), (18), (20) and (22) is 3, thus $k = 3$. Therefore, from theorem 1, the iterative methods defined by (17), (19), (21), and (23) have ninth order convergence.

4. Application of the New Iterative Methods

The present higher order methods given by (7), (17), (19), (21), and (23) are employed to solve nonlinear equations with multiple roots. To demonstrate the performance of the new higher order methods, we use ten particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the higher order methods for determining the multiple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in

the following tables.

Remark 1

The new ninth order method requires six function evaluations and has the order of convergence nine. To determine the efficiency index of the new method, we shall use the definition 2. Hence, the efficiency index of the new methods given by (17), (19), (21), and (23) is $\sqrt[9]{9}$ whereas the efficiency index of the eighth order methods (7) is given as $\sqrt[8]{8}$. We can see that the efficiency index of the new ninth order method is better than the eighth order method.

Remark 2

The test functions and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n for test functions with initial guess x_0 are displayed in tables. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. Furthermore, the computational orders of convergence (COC) are displayed in tables. From the tables we observe that the COC perfectly coincides with the theoretical result.

Table 1. Test functions and their roots

| Functions | m | Roots | Initial guess |
|--|-----------|---------------------------|---------------|
| $f_1(x) = (x^3 + x + 1)^m$ | $m = 10$ | $\alpha = -0.68232\dots$ | $x_0 = -0.8$ |
| $f_2(x) = (xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5)^m$ | $m = 5$ | $\alpha = -1.207647\dots$ | $x_0 = -1.3$ |
| $f_3(x) = ((x-1)^{10} - 1)^m$ | $m = 6$ | $\alpha = 0$ | $x_0 = -0.1$ |
| $f_4(x) = (\exp(x) + x - 20)^m$ | $m = 9$ | $\alpha = 2.842438\dots$ | $x_0 = 3.1$ |
| $f_5(x) = (\cos(x) + x)^m$ | $m = 15$ | $\alpha = -0.739085\dots$ | $x_0 = -0.9$ |
| $f_6(x) = (\sin(x)^2 - x^2 + 1)^m$ | $m = 66$ | $\alpha = 1.404491\dots$ | $x_0 = 1.6$ |
| $f_7(x) = (e^{-x^2} - e^{x^2} - x^8 + 10)^m$ | $m = 23$ | $\alpha = 1.239417\dots$ | $x_0 = 1.4$ |
| $f_8(x) = (6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1)^m$ | $m = 11$ | $\alpha = 0.257530\dots$ | $x_0 = -0.5$ |
| $f_9(x) = (\tan(x) - e^x - 1)^m$ | $m = 39$ | $\alpha = 1.371045\dots$ | $x_0 = 1.5$ |
| $f_{10}(x) = (\ln(x^2 + 3x + 5) - 2x + 7)^m$ | $m = 121$ | $\alpha = 5.469012\dots$ | $x_0 = 5.8$ |

Table 2. Comparison of iterative methods

| f_i | Newton (7) | Halley (19) | Osada (17) | Euler-Cheb (21) | Chun-Neta (23) |
|----------|-------------|-------------|-------------|-----------------|----------------|
| f_1 | 0.221e-614 | 0.358e-919 | 0.107e-776 | 0.497e-794 | 0.141e-880 |
| f_2 | 0.238e-443 | 0.519e-932 | 0.139e-563 | 0.283e-614 | 0.254e-698 |
| f_3 | 0.694e-328 | 0.138e-542 | 0.641e-301 | 0.762e-353 | 0.219e-448 |
| f_4 | 0.868e-395 | 0.745e-778 | 0.119e-479 | 0.263e-513 | 0.471e-669 |
| f_5 | 0.104e-760 | 0.314e-928 | 0.199e-879 | 0.118e-884 | 0.431e-921 |
| f_6 | 0.612e-319 | 0.522e-449 | 0.786e-371 | 0.183e-372 | 0.135e-445 |
| f_7 | 0.829e-240 | 0.552e-456 | 0.121e-301 | 0.746e-307 | 0.298e-436 |
| f_8 | 0.147e-166 | 0.261e-290 | 0.210e-188 | 0.316e-198 | 0.544e-262 |
| f_9 | 0.255e-189 | 0.314e-620 | 0.315e-189 | 0.906e-232 | 0.516e-453 |
| f_{10} | 0.759e-1216 | 0.148e-1386 | 0.144e-1370 | 0.719e-1371 | 0.319e-1386 |

Table 3. COC of various iterative methods

| f_i | Newton (7) | Halley (19) | Osada (17) | Euler-Cheb (21) | Chun-Neta (23) |
|----------|------------|-------------|------------|-----------------|----------------|
| f_1 | 7.9997 | 8.9999 | 9.0002 | 8.9999 | 9.0000 |
| f_2 | 8.0000 | 9.0004 | 8.9998 | 9.0001 | 8.9997 |
| f_3 | 8.0000 | 9.0006 | 9.0000 | 9.0000 | 8.9999 |
| f_4 | 8.0000 | 8.9999 | 9.0004 | 8.9997 | 9.0001 |
| f_5 | 8.0001 | 9.0003 | 8.9998 | 8.9997 | 8.9997 |
| f_6 | 8.0000 | 8.9996 | 9.0000 | 9.0000 | 8.9998 |
| f_7 | 8.0001 | 8.9998 | 9.0000 | 9.0000 | 9.0000 |
| f_8 | 7.9998 | 9.0000 | 8.9993 | 8.9995 | 9.0000 |
| f_9 | 8.0001 | 8.9996 | 9.0002 | 9.0000 | 8.9996 |
| f_{10} | 7.9999 | 9.0001 | 9.0000 | 8.9999 | 8.9999 |

5. Conclusions

In this paper, we have introduced four new ninth order iterative methods for solving nonlinear equations with multiple roots. Convergence analysis proves that the new methods preserve their order of convergence. Simply repeating the iterative process, we have achieved a ninth order of convergence. The prime motive of presenting these new methods was to establish a higher order of convergence method than the existing methods [1-21]. We have examined the effectiveness of the new methods by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that

the convergence of the tested multipoint methods of the ninth order is remarkably fast. The main purpose of demonstrating the new methods for different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. We have shown numerically, and verified, that the new methods converge to the order nine. Empirically, we have found that the Halley ninth order method is performing better than the other similar methods. Finally, we have constructed new higher order methods, but unfortunately the efficiency of the methods has not increased, hence further investigation is essential.

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