

# A Family of $L(\alpha)$ – stable Block Methods for Stiff Ordinary Differential Equations

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**Abstract** A family of self-starting  $L(\alpha)$ -stable linear multistep block methods (LMBMs) for solving stiff initial value problems (IVPs) in ordinary differential equations (ODEs) is proposed. The constructions are done by pairing three-step Top Order Method (TOM) and  $k$ -step Backward Differentiation Formulas (BDF) and using the shifting techniques introduced by Ajie *et al* (2011). The performance of the resultant block methods on the numerical examples considered shows their effectiveness.

**Keywords** Top Order Method (TOM), Backward Differentiation Formulas (BDF),  $L(\alpha)$  – stable,  $A(\alpha)$  – stable, Linear multistep block methods (LMBMs)

## 1. Introduction

Many physical problems when modeled mathematically result in ordinary differential equations. Some of these equations don't have solution in closed form. Therefore there is need to provide good numerical methods to approximate their solutions. In this paper, we introduced a self-starting  $L(\alpha)$ -stable family of order 4, 5 and 6 for stiff initial value problem of the form

$$y' = f(t, y), y'(t_0) = y_0, t \in [t_0, T_n] \quad (1.1)$$

It is assumed that  $f$  is continuous and satisfies Lipschitz condition as specified in Henrici (1962). Solving (1.1) using  $k$ -step linear multistep methods require

The provision of  $k-1$  starting values

$A$ -stable method if it is stiff

High order method if high accuracy is needed.

Standard linear multistep methods suffer from Dahlquist order barrier theorem that no  $A$ -stable method can have order greater than two. Many steps have been taken by different numerical analyst to circumvent this barrier and construct high order  $A$ -stable schemes as can be seen in Brugnano and Trigiante (1998), Akinfenwa *et al* (2011), Ajie *et al* (2011).

Curtiss and Hirshfelder (1952) introduced backward differentiation formulas (BDFs) also called Gear's methods (because it was used extensively by Gear). Cases for  $k \leq 5$

played important role in many algorithms for solving stiff systems by initial value methods (IVMs).

Block methods were also introduced to both circumvent the barrier and provide the  $k-1$  starting values to  $k$ -step LMM. They have the capacity to generate simultaneously  $k$  approximate solutions. For block methods see the following references Akinfenwa (2011), Ajie *et al* (2011), Fatunla (1991, 1994), Zanariah and Suleiman (2007), Cash and Diamantakis (1994), Brugnano and Magherini (2000, 2007), Muka and Ikhile (2012), Brugnano and Trigiante (1997, 1998), Shampine and Watts (1972), Onumanyi *et al.* (1994, 2001), Bond and Cash (1979).

In the section that follows, we give the construction of the methods; section three gives the stability analysis. In section four, we give numerical examples and conclude in section five.

## 2. Construction of the Methods

The general linear multistep formula (LMF) is given by

$$\sum_{j=0}^k \alpha_j y_{n+j} - h_n \sum_{j=0}^k \beta_j f_{n+j} = 0 \quad (2.1)$$

where  $h_n$  is the variable step size,  $\alpha_j$  and  $\beta_j$ ,  $j = 0, 1, 2, \dots, k$  are coefficients which can be uniquely determined.

By introducing two polynomials  $\rho(z) = \sum_{j=0}^k \alpha_j z^j$ ,

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$\sigma(z) = \sum_{j=0}^k \beta_j z^j$  (2.1) can be rewritten as

$$\rho(z) - h\sigma(z) = 0 \quad (2.2)$$

Taking  $\sigma(z)$  in its simplest form,  $\sigma(z) = \beta_k z^k$  we obtain a class of order  $k$ ,  $k$ -step LMF known as BDF and are given by

$$\sum_{r=0}^k \alpha_r y_{n+r} = h_n \beta_k f(x_{n+k}, y_{n+k}) \quad (2.3)$$

A popular example is the implicit Euler method,  $y_{n+1} - y_n = h_n f_{n+1}$ , which is the only L-stable member of the BDF family known of order one. It is in fact known that BDF of order  $k \geq 7$  are zero unstable hence will not converge. Cases of  $k=4, 5$  and  $6$  will be used in our constructions.

Almost all the families of linear multistep methods are obtained by imposing one restriction or the other on the structure of the polynomials  $\rho(z)$  or  $\sigma(z)$ . The family constructed without 'a priori' restrictions on such polynomials by imposing maximum order on (2.1) in order to get the coefficients  $\alpha_j$  and  $\beta_j$   $j=1, 2, \dots, k$  is

called TOMs. This family was considered by Dahlquist (1956) and presented as unstable methods. Example is the Sixth Order TOM below which we will use in constructing our family of methods.

$$\begin{aligned} & \frac{1}{60} (11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n) \\ & = \frac{h_m}{20} (f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n) \end{aligned} \quad (2.4)$$

Their stability properties as boundary value methods (BVMs) were studied by Amodio (1996).

The objective of this paper is to derive efficient higher order block methods which are  $L(\alpha)$ -stable among block LMF for efficiency. Higher order methods are considered for  $p=4, 5, 6$  where  $p$  is the order. These are obtained by pairing TOMs with  $k=3$ , order  $p=2k$  and the BDF with  $k=4, 5, 6$  each of order  $k$ . The pairs are then shifted forwards simultaneously repeatedly  $(k-2)$  times to give a set of  $2(k-1)$  equations which can be solved to obtain the unknowns  $y_{n+1}, y_{n+2}, y_{n+3}, \dots, y_{n+2(k-1)}$ . For values of  $k=4, 5, 6$ , the construction yields the new one step block method of the form

$$A_{(1)} Y_{\omega+1} + A_{(0)} Y_{\omega} = h B_{(0)} F_{\omega} + h B_{(1)} F_{\omega+1} \quad (2.5)$$

where

$$\begin{aligned} Y_{\omega+1} &= (y_{n+1}, y_{n+2}, y_{n+3}, \dots, y_{n+k})^T; \quad Y_{\omega} = (y_{n-k+1}, \dots, y_{n-2}, y_{n-1}, y_n)^T; \\ F_{\omega+1} &= (f_{n+1}, f_{n+2}, f_{n+3}, \dots, f_{n+k})^T; \quad F_{\omega} = (f_{n-k+1}, \dots, f_{n-2}, f_{n-1}, f_n)^T \end{aligned} \quad (2.6)$$

Case  $k=4$

$$A_1 = \begin{pmatrix} -\frac{9}{20} & \frac{9}{20} & \frac{11}{60} & 0 & 0 & 0 \\ -\frac{4}{3} & 3 & -4 & \frac{25}{12} & 0 & 0 \\ -\frac{11}{60} & -\frac{9}{20} & \frac{9}{20} & \frac{11}{60} & 0 & 0 \\ \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12} & 0 \\ 0 & -\frac{11}{60} & -\frac{9}{20} & \frac{9}{20} & \frac{11}{60} & 0 \\ 0 & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12} \end{pmatrix}; \quad A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{11}{60} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad B_1 = \begin{pmatrix} \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{20} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case  $k=5$

[illegible]

[illegible]

$$B_1 = \begin{pmatrix} \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{20} & \frac{9}{20} & \frac{9}{20} & \frac{1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have constructed the one-step block LMF of order  $p = 4, 5, 6$  in this section using only a pair of LMF. The problem of looking for additional methods to form blocks are overcome.

### 3. Stability Analysis of the Methods (See Akinfenwa *et al.* (2011))

Applying (2.5) to the test equation

$$y' = \lambda y; \lambda < 0 \quad (3.1)$$

gives

$$Y_{\omega+1} = D(z)Y_{\omega}; z = \lambda h \quad (3.2)$$

where

$k = 4$

$$R(z) = \frac{550200 + 1409604z + 1527924z^2 + 814451z^3 + 100732z^4 + 258z^5}{550200 - 1891596z + 2973900z^2 - 2787421z^3 + 1681714z^4 - 492696z^5 + 40896z^6}$$

$k = 5$

$$R(z) = \frac{88665506760 + 327685244700z + 550638915600z^2 + 539051173775z^3 + 315211698819z^4 + 80428456890z^5 + 3461130810z^6 + 988200z^7}{(137 - 60z)^2(4724040 - 16195620z + 25744080z^2 - 24759905z^3 + 15626211z^4 - 6398694z^5 + 402570z^6)}$$

$k = 6$

$$R(z) = \frac{912452702280 + 4438612059420z + 10141284207180z^2 + 14269794921725z^3 + 13514390215592z^4 + 8649951148717z^5 + 3249267317378z^6 + 449637707960z^7 + 6952214800z^8 + 216000z^9}{(-49 + 20z)^3(-7755720 + 30332820z - 55691220z^2 + 63063875z^3 - 48434858z^4 + 25861577z^5 - 9172512z^6 + 352170z^7)}$$

$$D(z) = (A_{(1)} - zB_{(1)})^{-1}(A_{(0)} + zB_{(0)}) \quad (3.3)$$

From (3.3), we obtain the stability function  $R(z)$  which is a rational function of real coefficients given by

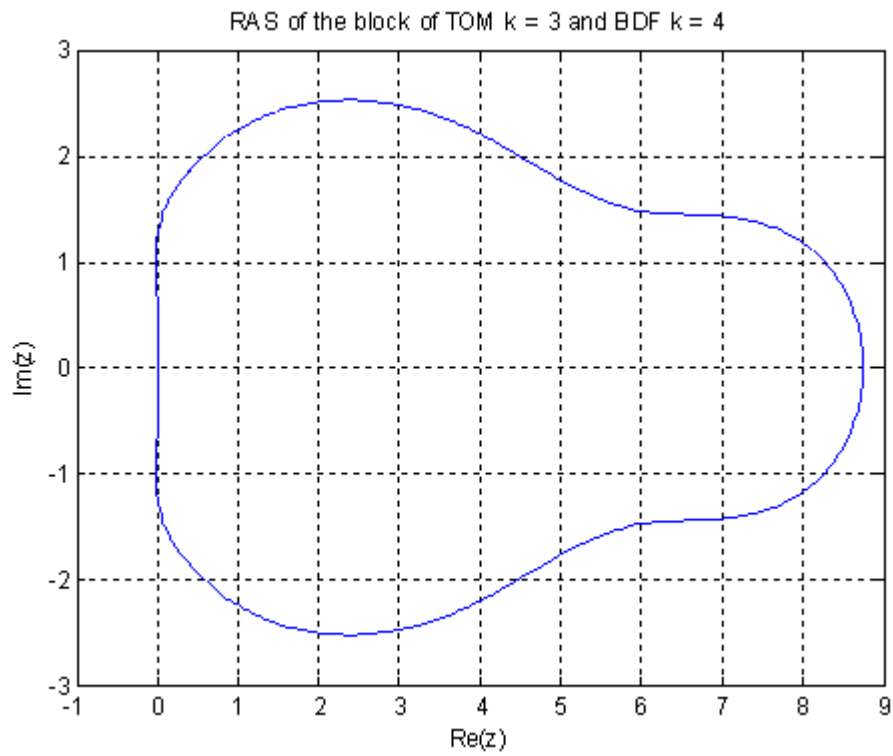
$$R(z) = \text{Det}[IR - D(z)] \quad (3.4)$$

where  $I$  is a  $k \times k$  identity matrix. The stability domain  $S$  of a one-step block method is

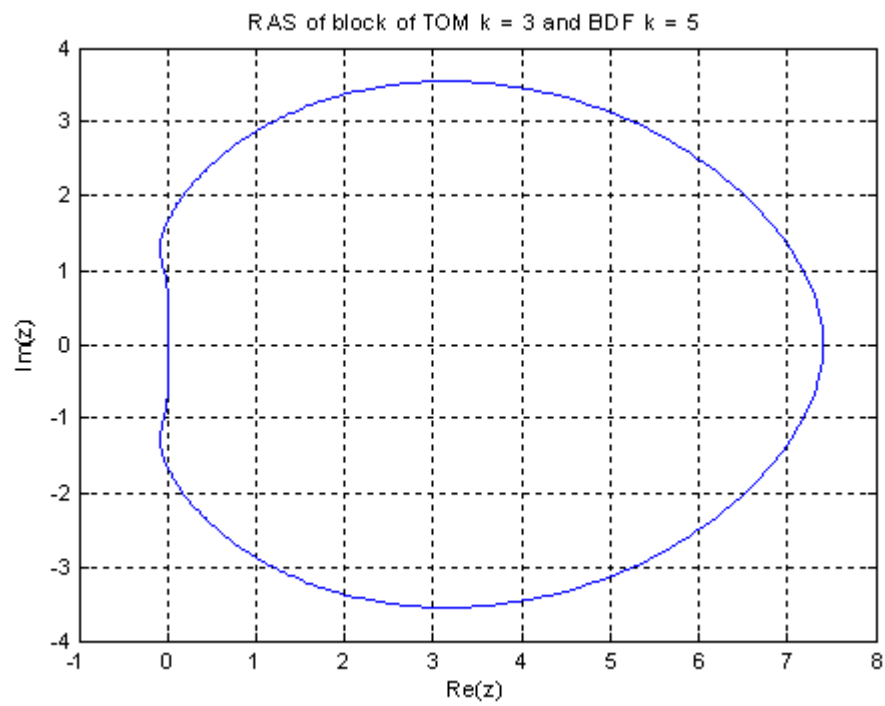
$$S = \{z \in \mathbb{C} : R(z) \leq 1\} \quad (3.5)$$

From the analysis given we easily obtain as follows the rational functions  $R(z)$  for  $k = 4, 5, 6$  which satisfies (3.5).

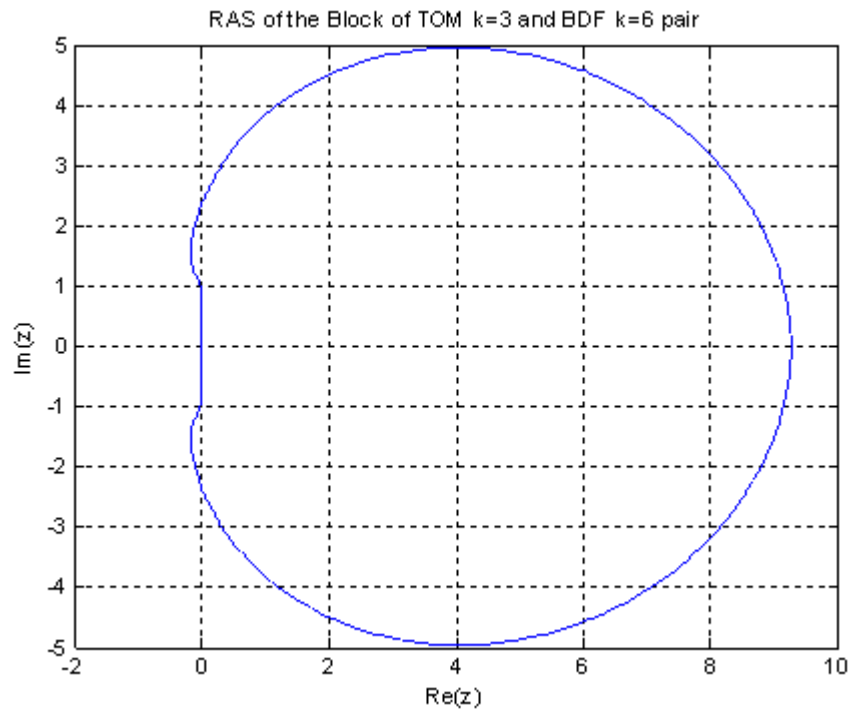
The boundary locus plots of the stability domain for  $k=4, 5$ , and  $6$  are given as follows:



This is said to be A-stable ( $A(90^\circ)$ -stable).



This is said to be  $A(82^\circ)$ -stable.



This is said to be  $A(81^\circ)$ -stable.

Hence we conclude the stability analysis of the three new methods in section 3.0 to be

$L(\alpha)$ -stable since  $\rho(D(\infty)) = 0$ . (see Chartier (1993))

## 4. Numerical Experiments

### Example 4.1

The test equation  $y' = -10y$  is solved with  $h = 0.01$  using the three methods, order 4, 5 and 6 and the results displayed in the table below

X	Errors in order 4	Errors in order 5	Errors in order 6
0.02	1.40e-006	1.06e-007	7.93e-009
0.04	1.46e-006	1.02e-007	7.36e-009
0.06	-7.66e-007	1.19e-007	6.83e-009
0.08	1.39e-007	3.15e-007	-7.023e-009
0.10	2.86e-007	3.06e-007	-2.12e-008
0.12	-8.41e-007	2.569e-007	-1.45e-008
0.14	-2.67e-007	2.26e-007	-1.15e-008

It can easily be seen that the accuracy increases as the order increases.

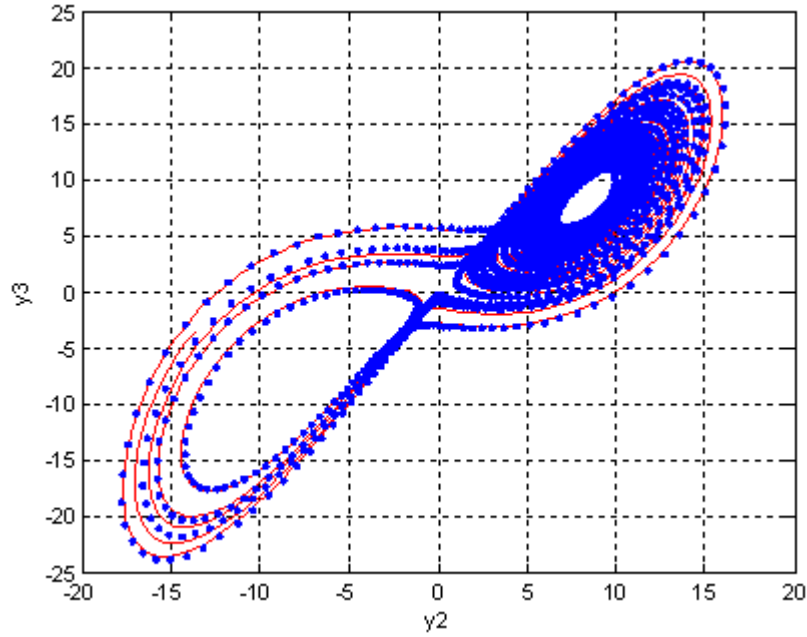
**Example 4.2** Lorenz chaotic attractor (see Sparrow (1982))

$$y' = \begin{bmatrix} -\beta & 0 & y_2 \\ 0 & -\sigma & \sigma \\ -y_2 & \rho & -1 \end{bmatrix} y; \quad y_c = \begin{bmatrix} \rho-1 \\ \eta \\ \eta \end{bmatrix}; \quad y_0 = y_c + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\rho = 28; \quad \sigma = 10; \quad \beta = \frac{8}{3}; \quad \eta = \sqrt{\beta(\rho-1)};$$

The above Lorenz equation is solved using order 6 of our method and step size  $h = 0.01$ . The result is compared with the

one computed using ode45 and are displayed in the figure below

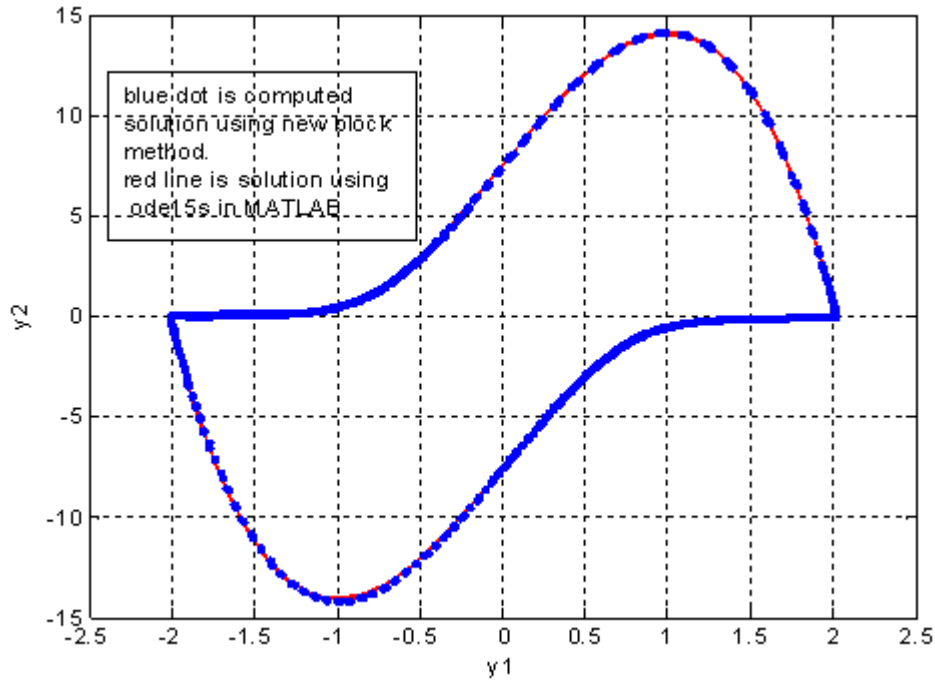


The blue dot is the computed results using our new block method while the red line is the one computed with ode45.

#### Example 4.3

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + \mu y_2(1 - y_1^2); \quad y_1(0) = 2, \quad y_2(0) = 0 \end{aligned}$$

The Vander Pol problem is used to demonstrate the ability of the methods to solve stiff nonlinear problems. The above Vander Pol is solved for  $\mu = 10$ , using order 6 of our method and step size  $h = 0.01$ . The result is displayed in the figure below



## 5. Conclusions

A new family of  $L(\alpha)$ -stable block methods that is self-starting has been introduced. They combine the accuracy of TOMs and the stability properties of BDFs in the block implicit pairs formed for adequate block-by-block forward integration. The analysis of the stability properties shows that the methods are  $L(\alpha)$ -stable for  $4 \leq k \leq 6$ . The numerical experiments considered showed that the methods are comparable to existing ones.

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