

A Study of Solutions to Euler Equations for a One Dimensional Unsteady Flow

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Abstract In this paper we deal with the Euler equations for Isothermal gas. In analyzing the equations we obtain two real and distinct eigenvalues which enables us to determine the wave structure of the possible solutions to the Riemann problem set up. By considering the Rankine-Hugoniot condition we obtain the shock wave solution analytically. The rarefaction wave solution is determined analytically by considering the fact that rarefaction wave lies along integral curves. To obtain the numerical solution to the Riemann problem that we set up, we use a relaxation scheme to discretize the Euler equations for isothermal gas. Finally we present the simulation results of the numerical solutions, that is, the approximate shock and rarefaction wave solutions are shown, graphically, and explained.

Keywords Isothermal Gas, Eigenvalues, Riemann Problem, Rankine-Hugoniot, Integral Curves, Relaxation Scheme

1. Introduction

Consider the Euler equations for isothermal gas. The system consists of Euler equations and is strictly hyperbolic with two real and distinct eigenvalues, whereby one is greater than the other. Depending on the initial data the eigenvalues may represent shock and rarefaction waves. The resolution of the discontinuities of a self-similar solution of the compressible Euler equation is sharper than the corresponding initial value solution, Ravi[4].

A shock tube problem is the study of the propagation of shock waves in a one dimensional tube. The energy of a shock wave dissipates relatively quickly within distance. Moving shocks are usually generated by the interaction of two bodies of gas at different pressure, with a shock wave propagating into the lower pressure gas and an expansion wave propagating into the higher pressure gas. The numerical computation of the shock tube problem by means on F wave digital principle showed that the MD Kichoffs network can successfully be extended by taking viscosity into account to represent the Navier Stokes equation Mengel[2].

A Riemann problem consists of equations together with the discontinuous initial data. The numerical dissipation of eight different schemes and five delimiters to numerical

computation of the Riemann problem compared to the resolution of discontinuities of each scheme were vital to improving the schemes accuracy and stability, Wang et al[8].

2. Mathematical Formulation of the Euler Equations

The Euler equations is derived from Newton's Second law of motion and is a system of conservation laws that can be written in the form

$$U_t + F(U)_x = 0 \quad (2.1)$$

Where U and $F(U)$ are the vectors of conserved variables and fluxes, given respectively by

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix}$$

Where ρ is density, p is pressure, u is the particle velocity and E is the total energy per unit volume.

Under the assumption that the entropy, s , is a constant everywhere, which is a simplification of the thermodynamics. Now the EOS is

$$P = p(\rho) = a^2 \rho \quad (2.2)$$

Where a is the wave propagation speed.

Under this assumption the energy equation becomes redundant and thus we have a 2X2 system. This makes equation 2.1 to be:

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$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad (2.3)$$

$$\text{Where } U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + \rho a^2 \end{bmatrix}$$

Thus we have,

$$\rho_t + u\rho_x + \rho u_x = 0 \quad (2.4)$$

$$u\rho_t + \rho u_t + 2\rho uu_x + u^2\rho_x + a^2\rho_x = 0 \quad (2.5)$$

Now using equation 2.5 we have

$$u(\rho_t + u\rho_x + \rho u_x) + \rho u_t + \rho uu_x + a^2\rho_x = 0$$

thus the two equations can be written as:

$$\rho_t + u\rho_x + \rho u_x = 0, \quad \rho u_t + \rho uu_x + a^2\rho_x = 0$$

which can be written in matrix form as:

$$\bar{u}_t + A(\bar{u})\bar{u}_x = 0$$

$$\text{Where } \bar{u} = \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad A(\bar{u}) = \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix}$$

Now to get the eigenvalues for the matrix $A(\bar{u})$, we use

the manipulation $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0, \text{ which gives rise to a}$$

quadratic equation with real roots

$$\begin{aligned} \lambda_1 &= u - a \\ \lambda_2 &= u + a \end{aligned} \quad (2.6)$$

Since the eigenvalues exist and are real values, thus the equation can be said to be hyperbolic. Now to get the eigenvectors for the corresponding eigenvalues we use $Ak_i = \lambda_i k_i$ where $i = 1, 2$

Thus using the first eigenvalue, λ_1 , we have:

$$K^{(1)} = \begin{bmatrix} 1 \\ -a/\rho \end{bmatrix}, \text{ as the corresponding eigenvector, while}$$

using the second eigenvalues, λ_2 , we have:

$$K^{(2)} = \begin{bmatrix} 1 \\ a/\rho \end{bmatrix} \text{ as the corresponding eigenvector.}$$

From the above eigenvalues and eigenvectors then we can be able to determine the structure of the waves that is by considering the Riemann's problem defined by

$$u(x, t = 0) = \begin{cases} u_L & x \leq 0 \\ u_R & x > 0 \end{cases}$$

The solution to this problem depends on the relative values of u_L and u_R . That is, for $u_L < u_R$, a rarefaction wave is going to develop while for $u_L > u_R$, a shock wave is going to appear. Thus for a case where no

vacuum is present the exact solutions will have the two different waves which would be associated with the eigenvalues 2.6.

3. Analytical Solution

Now to get the analytical solution for the shock wave for both the left and right shocks, we apply the Rankine-Hugoniot condition to the system that is,

$$Q_* - Q_R = s(\rho_* - \rho_R)$$

$$\left(\frac{Q^2}{\rho} + a^2\rho\right)_* - \left(\frac{Q^2}{\rho} + a^2\rho\right)_R = s(Q_* - Q_R)$$

This gives two equations with three unknowns, that is, ρ_* , Q_* and s , which can easily be solved for Q_* and s in terms of ρ_* , Leveque[3]

$$Q_* = \rho_* \frac{Q_R}{\rho_R} \pm a \sqrt{\frac{\rho_*}{\rho_R}} (\rho_* - \rho_R) \quad (3.1)$$

$$s = \frac{Q_R}{\rho_R} \pm a \sqrt{\frac{\rho_*}{\rho_R}} \quad (3.2)$$

In equations 3.1 and 3.2 it remains to express in terms of the known values ρ_R . For instance, in Leveque[3], we parametrize the curves by taking

$$\rho_{*i}(\xi; u_R) = \rho_R(1 + \xi) \quad i = 1, 2 \quad \text{and} \quad \xi = \frac{x}{t}$$

We then have

$$u_{*1}(\xi; u_R) = u_R + \xi \left[Q_R - a\rho_R \sqrt{1 + \xi} \right],$$

$$s_1(\xi; u_R) = \frac{Q_R}{\rho_R} - a\sqrt{1 + \xi} \quad \text{and}$$

$$u_{*2}(\xi; u_R) = u_R + \xi \left[Q_R + a\rho_R \sqrt{1 + \xi} \right],$$

$$s_2(\xi; u_R) = \frac{Q_R}{\rho_R} + a\sqrt{1 + \xi}$$

Also the rarefaction wave solution takes the form

$$u(x, t) = \begin{cases} u_L & x \leq \xi_1 t \\ u_F & \xi_1 t < x < \xi_2 t \\ u_R & x \geq \xi_2 t \end{cases}$$

Since the rarefactions lie along the integral curve, Leveque[3], we consider the following equation.

$$u'_F = \alpha(\xi) r_i(u_F(\xi))$$

$$\text{Where } \xi = \frac{x}{t}, \quad \alpha(\xi) = \frac{1}{\nabla r_i(u_F(\xi)) \cdot r_i(u_F(\xi))}$$

$$\xi_1 \leq \xi \leq \xi_2$$

Therefore,

$$u'_F(\xi) = \frac{r_i(u_F(\xi))}{\nabla \lambda_i(u_F(\xi)) \cdot r_i(u_F(\xi))} \quad \xi_1 \leq \xi \leq \xi_2$$

With initial data $u_F(\xi) = u_L$, where $\xi_1 = \lambda_i(u_L)$ and $\xi_2 = \lambda_i(u_R)$

Now, using the first eigenvalue and equation 3.1, we can be able to construct the 1-rarefactions for the equations, thus the system of ODE's takes the form

$$\begin{aligned} \rho(\xi) &= \rho_1 e^{-(\xi - \xi_1)/a} \\ m(\xi) &= \rho_1 (\xi - a) e^{-(\xi - \xi_1)/a} \end{aligned}$$

Where on eliminating ξ to solve for m as a function of ρ , we have

$$m(\rho) = \rho m_1 / \rho_1 - a \rho \log(\rho / \rho_1) \quad (3.3)$$

Similarly, using the second eigenvalue to construct the 2-rarefactions, we have

$$m(\rho) = \rho m_1 / \rho_1 + a \rho \log(\rho / \rho_1) \quad (3.4)$$

4. Numerical Solution Using the Relaxation Scheme

Now, for us to solve the Euler equation 2.3 numerically we will use the relaxation scheme, since it they are stable and conservative discretizations of the original conservation laws.

In equation 2.3 we had,

$$\begin{aligned} U_t + F(u)_x &= 0 \quad U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \\ F(u) &= \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix} \end{aligned}$$

$$\text{With initial data } U(x,0) = U_0(x) = \begin{cases} u_L & x \leq 0 \\ u_R & x > 0 \end{cases}$$

Now introducing the relaxation system, we have

$$\left. \begin{aligned} u_t + v_x &= 0 \quad u \in \mathbb{R}^2, v \in \mathbb{R}^2, x \in \mathbb{R}^1, t > 0 \\ v_t + Au_x &= -\frac{1}{\xi}(v - F(u)), \quad \xi > 0 \end{aligned} \right\} \quad (4.1)$$

$$\text{Where } A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \text{ is a positive diagonal matrix to}$$

be chosen.

For ξ sufficiently small, it is expected that by solving (4.1) properly, one can obtain good approximations to the original conservation laws

The positive constant a need satisfy: Shi J. *et al* [5],

$$-\sqrt{a} \leq F'(u) \leq \sqrt{a} \quad \forall u$$

For the relaxation system (4.1), the initial data is:

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \equiv F(u_0(x))$$

Now introducing the spatial grid points $x_{j+1/2}$ with mesh width $h_j = x_{j+1/2} - x_{j-1/2}$ while the discrete time level t_n are spaced uniformly with space step $k = t_{n+1} - t_n$ for $n=0, 1, 2$.

Here we denote by W_j^n the approximate cell average of a quantity W in the cell $[x_{j-1/2}, x_{j+1/2}]$ at time t_n and by $W_{j+1/2}^n$ the approximate point value of W at $x = x_{j+1/2}$ and $t = t_n$.

A spatial discretization to equation 4.1 in conservation form can be written as:

$$\left. \begin{aligned} \frac{\partial}{\partial t} u_j + \frac{1}{h_j} (v_{j+1/2} - v_{j-1/2}) &= 0, \\ \frac{\partial}{\partial t} v_j + \frac{1}{h_j} A(u_{j+1/2} - u_{j-1/2}) &= -\frac{1}{\xi} (v_j - F_j) \end{aligned} \right\} \quad (4.2)$$

Where the averaged quantity F_j is defined by

$$F_j = \frac{1}{h_j} \int_{x_{j-1/2}}^{x_{j+1/2}} F(u) dx \approx F(u_j)$$

Since the relaxation system 4.1 has two characteristic variables, see Shi J. *et al* 5,

$$v \pm A^{1/2} u \quad (4.3)$$

that travel with the frozen characteristic speeds $\pm A^{1/2}$ respectively.

For better accuracy we use a second-order scheme that is the Van Leer's MUSCL scheme, see Van Leer[7].

Applying this scheme to the p^{th} component that is equation 4.3, gives

$$\left. \begin{aligned} (v + \sqrt{a_p} u)_{j+1/2} &= (v + \sqrt{a_p} u)_j + \frac{1}{2} h_j \sigma_j^+, \\ (v - \sqrt{a_p} u)_{j+1/2} &= (v - \sqrt{a_p} u)_{j+1} - \frac{1}{2} h_j + \sigma_{j+1}^- \end{aligned} \right\} \quad (4.4)$$

Where σ_j is the slope of $v \pm \sqrt{a_p} u$ on the j -th cell which we define using Sweby's notation, see Sweby[6].

$$\begin{aligned} \sigma_j^\pm &= \frac{1}{h_j} (v_{j+1} \pm \sqrt{a_p} u_{j+1} - v_j \mp \sqrt{a_p} u_j) \phi(\theta_j^\pm), \\ \theta_j^\pm &= \frac{v_j \pm \sqrt{a_p} u_j - v_{j-1} \mp \sqrt{a_p} u_{j-1}}{v_{j+1} \pm \sqrt{a_p} u_{j+1} - v_j \mp \sqrt{a_p} u_j} \end{aligned}$$

Where ϕ is a slope-limiter function given as, Van Leer[7]

$$\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$$

Solving equations 4.4 for $u_{j+1/2}$ and $v_{j+1/2}$ gives

$$\left. \begin{aligned} u_{j+1/2} &= \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2\sqrt{a_p}}(v_{j+1} - v_j) + \frac{1}{4\sqrt{a_p}}(h_j\sigma_j^+ + h_{j+1}\sigma_{j+1}^-), \\ v_{j+1/2} &= \frac{1}{2}(v_j + v_{j+1}) - \frac{\sqrt{a_p}}{2}(u_{j+1} - u_j) + \frac{1}{4}(h_j\sigma_j^+ + h_{j+1}\sigma_{j+1}^-) \end{aligned} \right\} \quad (4.5)$$

Applying 4.5 in 4.4 we have,

$$\left. \begin{aligned} \frac{\partial}{\partial t} u + \frac{1}{2h_j}(v_{j+1} - v_{j-1}) - \frac{\sqrt{a_p}}{2h_j}(u_{j+1} - 2u_j + u_{j-1}) - \frac{1}{4h_j}(h_{j+1}\sigma_{j+1}^- - h_j(\sigma_j^+ + \sigma_{j-1}^-) + h_{j-1}\sigma_{j-1}^+) &= 0 \\ \frac{\partial}{\partial t} u + \frac{1}{2h_j}(u_{j+1} - u_{j-1}) - \frac{\sqrt{a_p}}{2h_j}(v_{j+1} - 2v_j + v_{j-1}) + \frac{\sqrt{a_p}}{4h_j}(h_{j+1}\sigma_{j+1}^- + h_j(\sigma_j^+ - \sigma_{j-1}^-) - h_{j-1}\sigma_{j-1}^+) &= -\frac{1}{\xi}(v_j - F^{(p)}(u_j)) \end{aligned} \right\} \quad (4.6)$$

Where $F^{(p)}$ is the p-th component of F.

Since the one dimensional Euler equation 2.3 has two eigen values $u \pm \hat{a}$ we take

$$a_1 = \sup|u - \hat{a}|, \quad a_2 = \sup|u + \hat{a}|$$

Now, to obtain the time discretization for the relaxation scheme we use a second-order TVD Runge-Kutta splitting scheme which was introduced by Jin, Shi J. et al[5]

$$\text{Denote } D + W_j = \frac{1}{h_j}(W_{j+1/2} - W_{j-1/2})$$

Then applying the second-order TVD Runge-Kutta splitting scheme to the time derivative in 4.6, that is applying it to 4.1 yields

$$\begin{aligned} u^* &= u^n, \\ v^* &= v^n + \frac{k}{\xi}(v^* - F(u^*)); \\ u^{(1)} &= u^* - kD + v^*, \\ v^{(1)} &= v^* - kAD + u^*, \\ u^{**} &= u^{(1)}, \\ v^{**} &= v^{(1)} - \frac{k}{\xi}(v^{**} - F(u^{**})) - 2\frac{k}{\xi}(v^* - F(u^*)); \\ u^{(2)} &= u^{**} - kD + v^{**}, \\ v^{(2)} &= v^{**} - kAD + u^{**}, \\ u^{n+1} &= \frac{1}{2}(u^n + u^{(2)}), \\ v^{n+1} &= \frac{1}{2}(v^n + v^{(2)}); \end{aligned}$$

5. Simulation

We use the value of h as 0.005 but choose the time step, k, according to Courant-Friedrichs-Lewy (CFL) condition.

$\frac{k}{h}|\lambda_{\max}| \leq \frac{1}{2}$ where λ_{\max} is the maximal (in absolute value) eigenvalue of the Jacobian matrix A given in section 2.6.

In the first case we consider the initial data $\rho_L = 0.9, \rho_R = 0.2 \quad v_L = 0.1, v_R = 0.2$, to yield:

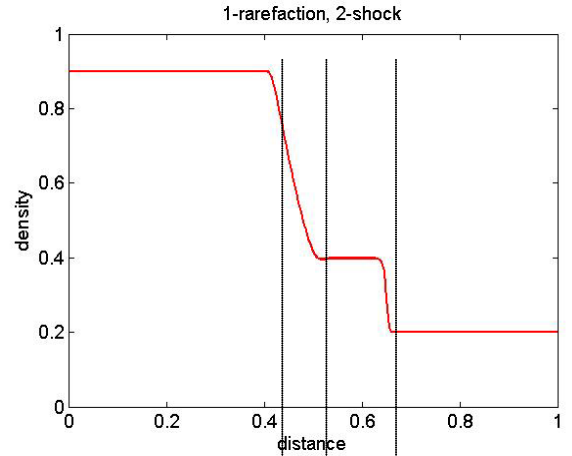


Figure 1.1.(a). Density profile for a 1-rarefaction followed by a 2-shock

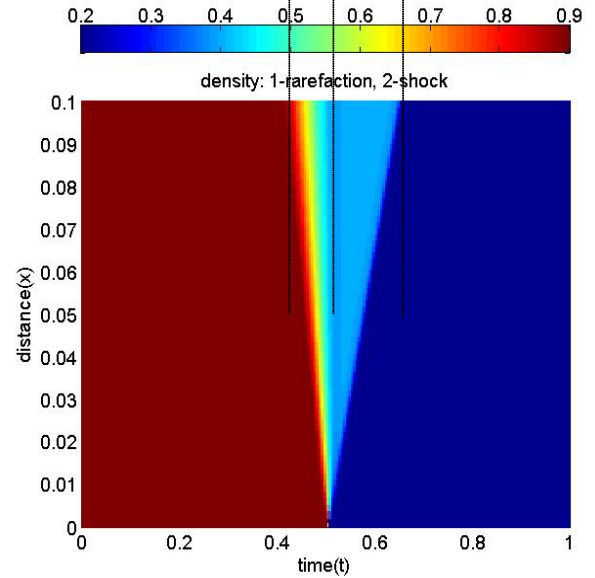


Figure 1.1.(b). distance-time graph of the density profile for a 1-rarefaction followed by a 2-shock

In the second case we consider the initial data $\rho_L = 0.2, \rho_R = 0.9 \quad v_L = 0.9, v_R = 0.5$, to yield:

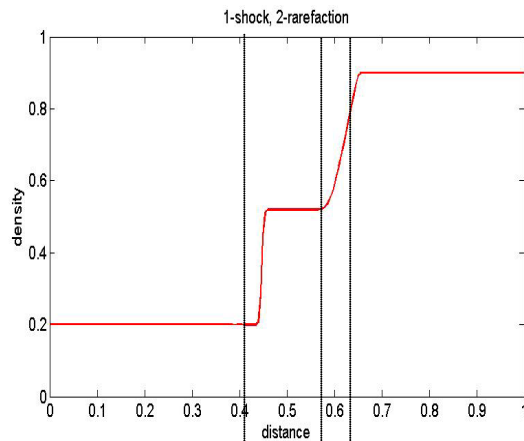


Figure 1.2(a). Density profile for a 1-shock followed by a 2-rarefaction.

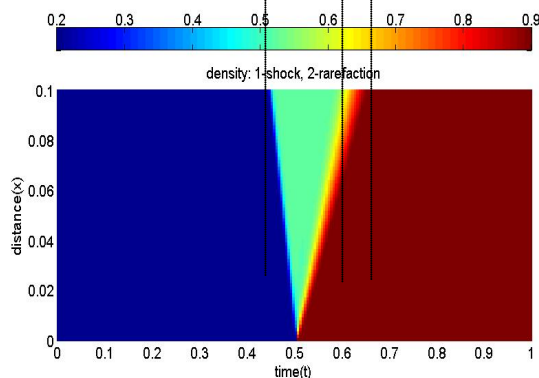


Figure 1.2(b). distance-time graph of the density profile for a 1-shock followed by a 2-rarefaction

6. Conclusions

Through the analysis of the Euler equation for an isothermal gas we were able to get two real and distinct eigenvalues which enabled us to determine the wave structures for the possible solutions to the Riemann problem. The analytical solutions was found by considering the Rankine-Hugoniot conditions to obtain the shock wave solution and using the fact that rarefaction waves lie on the integral curves.

The numerical scheme used is found to adequately approximate the shock wave as well as the rarefaction wave as evidenced by the simulations. In these simulations, two cases have been considered. In case 1 we considered some initial data which gave rise to a 1-rarefaction followed by a 2-shock waves, while in case 2 we considered another set of initial data which gave rise to a 1-shock followed by a

2-rarefaction waves.

Due to insufficient time the author of this study decided to deal with an isothermal case of the Euler equation and thus recommends for investigation of the Euler equations including the energy equation. We recommend for researchers to compare the relaxation scheme used in this research with other numerical schemes such as the Godunov scheme.

One can also check the accuracy of the numerical method used in this research by comparing it with the exact solution.

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REFERENCES

- [1] Eleuterio F. Toro. Riemann Solvers and Numerical Methods for Fluid Dynamic, Third edition, Springer, 2009.
- [2] Mengel A. (2006), "Numerical computation of the shock tube problem by means of wave digital principles," Advances in Radio Science, Volume 4, pp 161-164.
- [3] Randall J. Leveque. Numerical Methods for conservation laws, BirkhauserVerlag, 1992.
- [4] Ravi S. (1997) "Computational methods for self similar solutions of the compressible Euler Equations," Journal of Computational Physics, Volume 132, pp 327-345.
- [5] Shi Jin, Zhouping Xin (1995) "The relaxation schemes for systems of conservation laws in arbitrary space dimensions", John Wiley & Sons, Inc. pp 235-275
- [6] Sweby P.R, (1984) "High resolution schemes using flux limiters for hyperbolic conservation laws", SIAM J. Numer. Anal, volume 21, pp 995-1011
- [7] Van Leer B, (1979) "Toward the ultimate conservative difference schemes V.A second-order sequel to Godunov's method", J. Comput. Phys, volume 32, pp 101-136.
- [8] Wang Wen-Long, Li Huag and Pan Sha (2011) "Performance Comparison and Analysis of Different Schemes and Limiters," World Academy of Science, Engineering and Technology, Volume 79.