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A Note on Hypergeometric Functions of One and Two Variables

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Abstract The present paper deals with transformations of hypergeometric function of one variable and Kampe de Feriet hypergeometric function of two variables by using transformations between two terminating Saalschutzian $_4F_3(1)$ series.

Keywords Hypergeometric Functions, Transformation Formulae, Kampe de Feriet Hypergeometric Function

1. Introduction

Hypergeometric function is a beautiful tool of special function that plays an important role in the field of analysis. The transformation theory played a major role to provide a platform for the development of beautiful transformation. It is important to mentation that whenever generalized hypergeometric function reduces to a gamma function, the results are very important from application point of view in mathematics, statistics and mathematical physics. On account of usefulness, hypergeometric function have already explored to the same extent by a number of special function experts notably C. F. G auss, M. M. Kummer, S. Ramanujan, B. C. Berndt[1], G. N. Watson[5], W. N. Bailey[18], E. D. Rainville[2], L. J. Slater[9], H. Exton[6], G. Gasper[4], M. Rahman, G. E. Andrews[3], R. P. Agrawl[13], R. Y. Denis et al.[15], S. N. Singh[16], H. M. Srivastava et al.[7], R. P. Singh[14], M. A. Rakha et al.[10], Yong S. Kim et al.[19], Pankaj Srivastava[11], R. K. Saxena et al.[12], S. P. Singh[17], Kung-Yu Chen et al.[8] etc. In the present paper transformations of hypergeometric function of one and two variables established this paper should be useful from the application point of view.

2. Definitions and Notations

A generalized hypergeometric function is defined as

$$ {}_pF_q\left[\begin{matrix} a_1 & a_2 & ...a_p \\ b_1 & b_2 & ...b_q \end{matrix};x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_q)_n}\frac{x^n}{n!} $$

Where

$$ (a)_n=\begin{cases} 1, & if\ n=0, \\ a(a+1)(a+2)...(a+n-1), & if\ n\in N \end{cases} $$

The series converges when none of denominator parameter are zero or negative integer. It is converges for all x when p<q, for |x|<1, it is converges when p=q+1and for |x|<1 it is converges when Re(∑b_i-∑a_i)>0 and it is only converges for x=0 when x=0 when p>q+1 unless it reduces to a polynomial.

A Kampe de Feriet hypergeometric function of two variables is defined as,

$$ F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p) & (b_q) & (b_k') \\ (c_l) & (d_m) & (d_n') \end{matrix};x,y\right] $$

$$ =F\left[\begin{matrix} (a_p) & (b_q) & (b_k') \\ (c_l) & (d_m) & (d_n') \end{matrix};x,y\right] $$

$$ =\sum_{r,s=0}^{\infty}\frac{[(a_p)]_{r+s}[(b_q)]_r[(b_k')]_s}{[(c_l)]_{r+s}[(d_m)]_r[(d_n')]_s}\frac{x^r y^s}{r!s!} $$

Where (a_p) stands for the sequence of p parameters $a_1,\ldots a_p$ and |x|<1, |y|<1, l+m+n<p+q+k+1 for convergence.

We shall use the following known transformations and identity in our analysis as given in[1, 2, 4, 9]

$$ {}_3F_2\left[\begin{matrix} a, & b, & -n \\ c & d & \end{matrix};1\right]=\frac{(c-a)_n(d-a)_n}{(c)_n(d)_n} \tag{1} $$

$$ {}_3F_2\left[\begin{matrix} 1-c-d+a+b-n, & a, & -n \\ 1+\_a-c-n, & 1+a-d-n & \end{matrix};1\right] $$

([9], 2.5.11).

$$ {}_4F_3\left[\begin{matrix} a+c, & b+c, & d & -n \\ a+b+c, & e, & 1+c+d-e-n & \end{matrix};1\right] $$

$$ =\frac{(e-d)_n(e-c)_n}{(e)_n(e-d-c)_n}\,{}_4F_3\left[\begin{matrix} a, & b, & d, & -n \\ a+b+c, & e-c, & 1+d-e-n & \end{matrix};1\right] \tag{2} $$

([9], 2.4.1.7).

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$${}_4F_3\left[\begin{matrix}2a, & 2b, & c, & -n\\ 2c & a+b+\tfrac12 & a+b+\tfrac12-c-n & \end{matrix};1\right]=\frac{(a+\frac12)_n(c-a+\frac12)_n}{(a+b+\frac12)_n(c-a-b+\frac12)_n}\,{}_4F_3\left[\begin{matrix}b, & c-b, & \frac12-c-n & -n\\ c+\frac12 & \frac12-a-n, & \frac12+a-c-n & \end{matrix};1\right] \tag{3}$$

([9], 2.5.22).

$${}_4F_3\left[\begin{matrix}2a, & 2b, & c+\frac12 & -n\\ a+b+\frac12 & 2c & 1+a+b-c-n & \end{matrix};1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}\,{}_4F_3\left[\begin{matrix}a, & b, & \frac12+a+b-2c-n & -n\\ a+b+\frac12 & 1+a-c-n, & 1+b-c-n & \end{matrix};1\right] \tag{4}$$

([9], 2.5.25).

$${}_3F_2\left[\begin{matrix}a, & b, & -n\\ c, & d & \end{matrix};1\right]=\frac{(d-b)_n}{(d)_n}\,{}_3F_2\left[\begin{matrix}c-a, & b, & -n\\ c, & 1+b-d-n & \end{matrix};1\right] \tag{5}$$

([4], 3.1.1).

$${}_4F_3\left[\begin{matrix}a, & b, & c, & -n\\ d, & e, & f & \end{matrix};1\right]=\frac{(e-c)_n(f-c)_n}{(e)_n(f)_n}\,{}_4F_3\left[\begin{matrix}d-a, & d-b, & c, & -n\\ 1-e+c-n, & 1-f+c-n, & d & \end{matrix};1\right] \tag{6}$$

([1], Ch.10, eq. 6.3).

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(k,n+k) \tag{7}$$

([2], Ch.4, lemma 10(2)).

3. Main Result

In this section we shall establish following results.

Theorem 3.1:

$$\sum_{n,m=0}^{\infty}\frac{z^n}{n!}\frac{(a)_m(b)_m(-z)^m}{(c)_m(d)_m m!}(n+m)!\Omega_{n+m}=\sum_{n,m=0}^{\infty}\frac{(c-a)_n(d-a)_n z^n}{(c+d-a-b)_n n!}\frac{(a)_m z^m}{m!}\frac{(c+d-a-b)_{n+m}(n+m)!}{(c)_{n+m}(d)_{n+m}}\Omega_{n+m} \tag{i}$$

$$e^z\,{}_2F_2\left[\begin{matrix}a, & b\\ c, & d\end{matrix};-z\right]=F\left[\begin{matrix}c+d-a-b & & : & c-a, & d-a & ; & a\\ c, & d & : & c+d-a-b & & ; & -\end{matrix};z,z\right] \tag{ii}$$

$$(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}c-a, & d-a\\ c+d-a-b & \end{matrix};z\right]=F\left[\begin{matrix}c, & d & : & - & ; & a, & b\\ c+d-a-b & & : & - & ; & c, & d\end{matrix};z,-z\right] \tag{iii}$$

Theorem 3.2:

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{(e-d)_n z^n}{n!}\frac{(a)_m(b)_m(d)_m z^m}{(a+b+c)_m(e-c)_m m!}\frac{(n+m)!}{(e-d)_{n+m}}\Omega_{n+m}\\ &=\sum_{n,m=0}^{\infty}\frac{(e-c-d)_n z^n}{n!}\frac{(a+c)_m(b+c)_m(d)_m z^m}{(a+b+c)_m(e)_m m!}\times\frac{(e)_{n+m}(n+m)!}{(e-c)_m(e-d)_m}\Omega_{n+m}\end{aligned} \tag{i}$$

$$(1-z)^{d-e}\,{}_3F_2\left[\begin{matrix}a, & b, & d\\ a+b+c, & e-c & \end{matrix};z\right]=F\left[\begin{matrix}e & : & e-c-d & ; & a+c, & b+c, & d\\ e-c & : & & ; & a+b+c, & e & \end{matrix};z,z\right] \tag{ii}$$

$$(1-z)^{c+d-e}\,{}_3F_2\left[\begin{matrix}a+c, & b+c, & d\\ a+b+c, & e & \end{matrix};z\right]=F\left[\begin{matrix}e-c & : & e-d & ; & a, & b, & d\\ e & : & - & ; & a+b+c, & e-c & \end{matrix};z,z\right] \tag{iii}$$

Theorem 3.3:

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{(c-a-b+\frac12)_n z^n}{n!}\frac{(2a)_m(2b)\ (c)_m z^m}{(a+b+\frac12)_m(2c)_m m!}\frac{(n+m)!}{(c-a-b+\frac12)_{n+m}}\Omega_{n+m}\\ &=\sum_{n,m=0}^{\infty}\frac{(a+\frac12)_n(c-a+\frac12)_n z^n}{(c+\frac12)_n n!}\frac{(c-b)_m(b)_m z^m}{(c+\frac12)_m m!}\\ &\qquad\frac{(c+\frac12)_{n+m}(n+m)!}{(a+b+\frac12)_{n+m}(c-a-b+\frac12)_{n+m}}\Omega_{n+m}\end{aligned} \tag{i}$$

$$(1-z)^{a+b-c-\frac{1}{2}}\,{}_3F_2\left[\begin{matrix}2a, & 2b, & c\\ 2c, & a+b+\frac{1}{2} & \end{matrix};z\right]=F\left[\begin{matrix}c+\frac{1}{2} & : & a+\frac{1}{2} & c-a+\frac{1}{2} & ; & b, & c-b\\ a+b+\frac{1}{2} & : & c+\frac{1}{2} & & ; & c+\frac{1}{2} & \end{matrix};z,z\right] \tag{ii}$$

$${}_2F_1\left[\begin{matrix}b, & c-b\\ c+\frac{1}{2} & \end{matrix};z\right]{}_2F_1\left[\begin{matrix}a+\frac{1}{2}, & c-a+\frac{1}{2}\\ c+\frac{1}{2} & \end{matrix};z\right]=F\left[\begin{matrix}a+b+\frac{1}{2} & : & c-a-b+\frac{1}{2} & ; & 2a, & 2b, & c\\ c+\frac{1}{2} & : & - & ; & 2c, & a+b+\frac{1}{2} & \end{matrix};z,z\right] \tag{iii}$$

Theorem 3.4:

$$\sum_{n,m=0}^{\infty}\frac{(c-a-b)_n z^n}{n!}\frac{(2a)_m(2b)\ (c+\frac{1}{2})_m z^m}{(a+b+\frac{1}{2})_m(2c)_m m!}\frac{(n+m)!}{(c-a-b)_{n+m}}\Omega_{n+m}$$
$$=\sum_{n,m=0}^{\infty}\frac{(c-a)_n(c-b)_n z^n}{(2c-a-b+\frac{1}{2})_n n!}\frac{(a)_m(b)_m z^m}{(a+b+\frac{1}{2})_m m!}\frac{(2c-a-b+\frac{1}{2})_{n+m}(n+m)!}{(c)_{n+m}(c-a-b)_{n+m}}\Omega_{n+m} \tag{i}$$

$$(1-z)^{a+b-c}\,{}_3F_2\left[\begin{matrix}2a, & 2b, & c+\frac{1}{2}\\ a+b+\frac{1}{2}, & 2c & \end{matrix};z\right]=F\left[\begin{matrix}2c-a-b+\frac{1}{2} & : & c-a, & c-b & ; & a, & b\\ c & : & 2c-a-b+\frac{1}{2} & & ; & a+b+\frac{1}{2} & \end{matrix};z,z\right] \tag{ii}$$

$${}_2F_1\left[\begin{matrix}c-a, & c-b\\ 2c-a-b+\frac{1}{2} & \end{matrix};z\right]{}_2F_1\left[\begin{matrix}a, & b\\ a+b+\frac{1}{2} & \end{matrix};z\right]=F\left[\begin{matrix}c & : & 2a & 2b & c+\frac{1}{2} & ; & c-a-b\\ 2c-a-b+\frac{1}{2} & : & a+b+\frac{1}{2} & 2c & , & ; & -\end{matrix};z,z\right] \tag{iii}$$

Theorem 3.5:

$$\sum_{n,m=0}^{\infty}\frac{z^n}{n!}\frac{(a)_m(b)_m(-z)^m}{(c)_m(d)_m m!}(n+m)!\Omega_{n+m}=\sum_{n,m=0}^{\infty}\frac{(d-b)_n z^n}{n!}\frac{(c-a)_m(b)_m z^m}{m!}\frac{(n+m)!}{(c)_{n+m}(d)_{n+m}}\Omega_{n+m} \tag{i}$$

$$e^z\,{}_2F_2\left[\begin{matrix}a, & b\\ c, & d\end{matrix};-z\right]=F\left[\begin{matrix}- & : & c-a, & b & ; & d-b\\ d & : & c & & ; & -\end{matrix};z,z\right] \tag{ii}$$

$$(1-z)^{b-d}\,{}_2F_1\left[\begin{matrix}c-a, & b\\ c & \end{matrix};z\right]=F\left[\begin{matrix}d & : & a, & b & ; & -\\ - & : & c, & d & ; & -\end{matrix};z,-z\right] \tag{iii}$$

Theorem 3.6:

$$\sum_{n,m=0}^{\infty}\frac{z^n}{n!}\frac{(a)_m(b)_m(c)_m(-z)^m}{(d)_m(e)_m(f)_m m!}(n+m)!\Omega_{n+m}=\sum_{n,m=0}^{\infty}\frac{(e-c)_n(f-c)_n z^n}{n!}\frac{(d-a)_m(d-b)_m(c)_m(-z)^m}{(d)_m m!}$$
$$\frac{(n+m)!}{(e)_{n+m}(f)_{n+m}}\Omega_{n+m} \tag{i}$$

$$e^z\,{}_3F_3\left[\begin{matrix}a, & b, & c\\ d, & e, & f\end{matrix};z\right]=F\left[\begin{matrix}- & & : & e-c, & f-c & ; & d-a, & d-b, & c\\ e, & f & : & & & ; & d & & \end{matrix};z,-z\right] \tag{ii}$$

$${}_2F_0\left[\begin{matrix}e-c, & f-c\\ - & \end{matrix};z\right]{}_3F_1\left[\begin{matrix}d-a, & d-b, & c\\ d & & \end{matrix};-z\right]=F\left[\begin{matrix}e, & f & : & - & ; & a, & b, & c\\ - & & : & - & ; & d, & e, & f\end{matrix};z,-z\right] \tag{iii}$$

As an illustration we give proof of theorem (3.1).

Let Ω_n is an arbitrary sequence of complex numbers, multiplying both sides of (1) by $z^n\Omega_n$ and summing over n from 0 to ∞ we get,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(a)_m(b)_m(-n)_m}{(c)_m(d)_m m!}z^n\Omega_n=\sum_{n=0}^{\infty}\frac{(c-a)_n(d-a)_n}{(c)_n(d)_n}\sum_{m=0}^{n}\frac{(1-c-d+a+b-n)_m(a)_m(-n)_m}{(1+a-c-n)_m(1+a-d-n)_m m!}z^n\Omega_n$$

Making use of the identity (7) and (a-n)n=(-1)n(1-a)n in above we get after some simplification,

$$\sum_{n,m=0}^{\infty}\frac{z^n}{n!}\frac{(a)_m(b)_m(-z)^m}{(c)_m(d)_m m!}(n+m)!\Omega_{n+m}=\sum_{n,m=0}^{\infty}\frac{(c-a)_n(d-a)_n z^n}{(c+d-a-b)_n n!}\frac{(a)_m z^m}{m!}\frac{(c+d-a-b)_{n+m}(n+m)!}{(c)_{n+m}(d)_{n+m}}\Omega_{n+m} \tag{i}$$

Equation (I) holds provided that both sides do exist. In particular, choosing $\Omega_n=\frac{1}{n!}$ and after some simplification we get,

$$e^z\,{}_2F_2\left[\begin{matrix}a, & b\\ c, & d\end{matrix};-z\right]=F\left[\begin{matrix}c+d-a-b & & : & c-a, & d-a & ; & a\\ c, & d & : & c+d-a-b & & ; & -\end{matrix};z,z\right] \tag{ii}$$

While choosing $\Omega_n = \dfrac{(c)_n (d)_n}{(c+d-a-b)_n n!}$ in eq. (I) and after some simplification we get,

$$(1-z)^{-a}\,{}_2F_1\left[\begin{matrix} c-a, & d-a \\ c+d-a-b & \end{matrix};z\right] = F\left[\begin{matrix} c, & d & : & - & ; & a, & b \\ c+d-a-b & & : & - & ; & c, & d \end{matrix};z,-z\right] \qquad \text{(iii)}$$

Similarly one can establish theorems (3.2), (3.3), (3.4), (3.5) and (3.6) by suitable transformation from (2) to (6) and appropriate selection of Ω_n.

4. Conclusions

The transformations of hypergeometric function of one and two variables established in this paper should be useful from the application point of view.

REFERENCES

[1] B. C. Bberndt, Ramanujan's Notebooks Part II, Springer-Verlag, New York, 1989.

[2] E. D. Rainville, Special Functions, Chelsea Publishing Company, Bronx, New York, 1960.

[3] G. E. Andrws, R. Askey and R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications Cambridge; Cambridge University Press, 1999, vol.71.

[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, vol. 96 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, UK, 2nd edition, 2004.

[5] G. N. Watson, The cubic transformation of the hypergeometric function, Quart. Math. 41(1910), 70-79.

[6] H. Exton, On the reducibility of the Kampe de Feriet function, Journal of Computational and Applied Mathematics vol.83, no.1 pp.119-121, 1997.

[7] H. M. Srivastava, M. I. Qureshi, R. Singh, and A. Arora, A family of hypergeometric integrals associated with Ramanujan's integral formula, Adv. Stud. Contemp. Math. 18 (2009), 113-125.

[8] Kung-Yu Chen, Shuoh-Jung Liu, H. M. Srivastava, Some double-series identities and associated generating-function relationships, Applied Mathematics Letters, 19 (2006) 887-893.

[9] L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cam- bridge, London and New York, 1966.

[10] M. A. Rakha, Adel K. Ibrahim, and Arjun K. Rathie, On the computations of contiguous relations for 2F1 hypergeometric series, Commun. Korean Math. Soc. 24 (2009), No.2, pp.291-302.

[11] Pankaj Srivastava, Certain transformations of generalized Kampe de Feriet function Vijnana Parishad Anusandan Patrika. Vol.45, No.4, (2002), pp.383-392.

[12] R. K. Saxena, C. Ram and N. Dudi, Some results associated with a generalized hyper- geometric function, Bulletin of Pure and Applied Sciences. Vol.24 (No.2) 2005, 305-316.

[13] R. P. Agarwal, Special function and their application, Keynote address, Proceedings of National Symposium on special Function and their application (Gorakhpur 1986).

[14] R. P. Singh, A note on double transformation of certain hypergeometric functions, Proceedings Edinburgh Math. Soc. (2), 14(1956), 221-227.

[15] R. Y. Denis and S. N. Singh, Hypergeometric Functions and Continued Fractions, Far East J. Math. Sci. 2(3) (2000), 385-400.

[16] S. N. Singh and S. P. Singh, Transformation of Kampe De Feriet function, Indian J. of Pure and Appl. Math. 13(10), October 1982.

[17] S. P. Singh, Certain theorems involving ordinary hypergeometric functions of one and two variables, South East Asian J. Math. And Mathematical. Soc. Vol.3 no.3 (2005), pp.59-63.

[18] W. N. Bailey, Generalized hypergeometric series, Cambridge University Press, Cambridge (1935), reprinted by Stechert-Hafner, New York (1964).

[19] Yong S. Kim, Tibor K. Pogany and Arjun K. Rathi, On a summation formula for the clausen's series 3F2 with applications Miskolc Mathematical Notes, Vol.10 (2009), No.2, pp.145-153.