

A New Family of Multipoint Iterative Methods for Finding Multiple Roots of Nonlinear Equations

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Abstract In this paper, we present a new family of multipoint iterative methods for finding multiple zeros of nonlinear equations. Per iteration the new method requires three evaluations of functions and one of its first derivative. We have analysed and proved the order of convergence of the new methods. Finally, the numerical examples demonstrate that the proposed methods are superior to the existing methods.

Keywords Modified Newton Method, Root-Finding, Nonlinear Equations, Multiple Roots, Order of Convergence, Efficiency Index

1. Introduction

Finding the roots of nonlinear equations is very important in numerical analysis and has many applications in engineering and other applied sciences. In this paper, we consider iterative methods to find a multiple root α of multiplicity m , i.e., $f^{(j)}(\alpha) = 0$, $j = 0, 1, \dots, m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation

$$f(x) = 0. \quad (1)$$

where $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function on an open interval I and it is sufficiently smooth in the neighbourhood of α . In recent years, some modifications of the Newton method for multiple roots have been proposed and analysed [2-12]. However, there are not many methods known to handle the case of multiple roots. Hence we present two new iterative methods of higher order for finding multiple zeros of a nonlinear equation and only use four evaluations of the function per iteration. In addition, the new methods have a better efficiency index than the third and fourth order methods given in [2,3-12]. In view of this fact, the new methods are significantly better when compared with the established methods. Consequently, we have found that the new methods are efficient and robust.

The well-known Newton's method for finding multiple roots is given by

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (2)$$

which converges quadratically [1,10]. For the purpose of this paper, we use (2) to construct new multipoint iterative methods for finding multiple roots of a nonlinear equation.

2. Construction of the Methods and Convergence Analysis

In this section we define two new iterative methods. In order to establish the order of convergence of the new methods we state the three essential definitions.

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0 \quad (3)$$

where ζ is the asymptotic error constant and $p \in \mathbb{R}^+$.

Definition 2 Let k be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [1,10] and defined as

$$p^{1/k} \quad (4)$$

where p is the order of the method.

Definition 3 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence [11] may be approximated by

$$\text{COC} \approx \frac{\ln |(x_{n+1} - \alpha)(x_n - \alpha)^{-1}|}{\ln |(x_n - \alpha)(x_{n-1} - \alpha)^{-1}|}. \quad (5)$$

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where $n \in \mathbb{N}$.

Recently Thukral [7] presented a fourth-order method for finding multiple roots of a nonlinear equation, which is given as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (6)$$

$$x_{n+1} = y_n - mbc \left(\frac{1+abc}{1+(a-2)bc} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (7)$$

$$\text{where } b = \frac{f^m(y_n)}{f^m(x_n)}, \quad c = \frac{f^{m-1}(x_n)}{f^{m-1}(y_n)}, \quad n \in \mathbb{N},$$

$a \in \mathbb{R}$, x_0 is the initial value and provided that the denominator of (6) and (7) are not equal to zero.

In fact, the new iterative methods presented in this paper are the extension of the above scheme. To develop the higher order iterative methods we use the above two steps and introduce the third step with a new parameter. First we define sixth-order iterative method and then followed by fifth-order iterative method.

2.1. The New Sixth-order Iterative Method

The new sixth-order method for finding multiple root of a nonlinear equation is expressed as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (8)$$

$$z_n = y_n - mbc \left(\frac{1+abc}{1+(a-2)bc} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (9)$$

$$x_{n+1} = z_n - mst \left(\frac{5-2a-(2-8a+2a^2)bc+(1+4a)(bc)^2}{5-2a-(12-12a+2a^2)bc} \right) \times \left(\frac{f(x_n)}{f'(x_n)} \right) \quad (10)$$

where

$$b = \frac{f^m(y_n)}{f^m(x_n)} \quad c = \frac{f^{m-1}(x_n)}{f^{m-1}(y_n)} \quad (11)$$

$$s = \frac{f^m(z_n)}{f^m(x_n)} \quad t = \frac{f^{m-1}(x_n)}{f^{m-1}(z_n)}$$

and $n \in \mathbb{N}$, $a \in \mathbb{R}$, x_0 is the initial value.

Theorem 1

Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently smooth function defined on an open interval D , enclosing a multiple zero of $f(x)$, with multiplicity $m > 1$. Then the family of iterative methods defined by scheme (10) has sixth-order convergence.

Proof

Let α be a multiple root of multiplicity m of a sufficiently smooth function $f(x)$, $e = x - \alpha$ and $\hat{e} = y - \alpha$, where y is defined in (8).

Using Taylor expansion of $f(x)$ and $f(y)$ about α , we have

$$f(x_n) = \left(\frac{f^{(m)}(\alpha)}{m!} \right) e_n^m \left[1 + c_1 e_n + c_2 e_n^2 + \dots \right] \quad (12)$$

$$f'(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!} \right) e_n^{m-1} \left[1 + \left(\frac{m+1}{m} \right) c_1 e_n + \dots \right] \quad (13)$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{m! f^{(m+k)}(\alpha)}{(m+k)! f^{(m)}(\alpha)}. \quad (14)$$

Moreover by (8), we have

$$\hat{e}_n = e_n - m \frac{f(x_n)}{f'(x_n)} = \frac{c_1}{m} e_n^2 + \dots \quad (15)$$

The expansion of $f(y_n)$ about α is given as

$$f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!} \right) \hat{e}_n^m \left[1 + c_1 \hat{e}_n + c_2 \hat{e}_n^2 + \dots \right]. \quad (16)$$

Simplifying (16), we get

$$f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!} \right) \left(\frac{c_1}{m} \right)^m \times e_n^{2m} \left[1 + \left(\frac{2mc_2 - (m+1)c_1^2}{c_1} \right) e_n + \dots \right] \quad (17)$$

Furthermore, we have

$$b = \left(\frac{c_1}{m} \right)^m e_n^m \left[1 + \left(\frac{2mc_2 - (m+2)c_1^2}{c_1} \right) e_n + \dots \right] \quad (18)$$

Since from (9) and (10), we have

$$z_n = \hat{e}_n - mbc \left(\frac{1+abc}{1+(a-2)bc} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (19)$$

$$e_{n+1} = z_n - mst \left(\frac{5-2a-(2-8a+2a^2)bc+(1+4a)(bc)^2}{5-2a-(12-12a+2a^2)bc} \right) \times \left(\frac{f(x_n)}{f'(x_n)} \right). \quad (20)$$

Substituting appropriate expressions in (10) and after simplification we obtain the error equation.

$$e_{n+1} = m^{-5} c_1^5 (1+2a)^2 e_n^6 + \dots \quad (21)$$

The error equation (21) establishes the fourth-order convergence of the new method defined by (10).

2.2. The New Fifth-Order Iterative Method

The new fifth-order method for finding multiple root of a nonlinear equation is expressed as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (22)$$

$$z_n = y_n - mbc \left(\frac{1+abc}{1+(a-2)bc} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (23)$$

$$x_{n+1} = z_n - mst \left(\frac{f(x_n)}{f'(x_n)} \right). \quad (24)$$

where b, c, s, t are given in (11), $n \in \mathbb{N}$, $a \in \mathbb{R}$, x_0 is the initial value and provided that the denominator of (22)-(24) are not equal to zero.

Theorem 2

Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently smooth function defined on an open interval D , enclosing a multiple zero of $f(x)$, with multiplicity $m > 1$. Then the family of iterative methods defined by scheme (24) has fifth-order convergence.

Proof

Substituting appropriate expressions in (24) and after simplification we obtain the error equation

$$e_{n+1} = 2m^{-4} c_1^4 (1+2a) e_n^5 + \dots \quad (25)$$

The error equation (25) establishes the fifth-order convergence of the new method defined by (24).

3. The Established Methods

For the purpose of comparison, we consider one sixth-order methods and two fifth-order methods presented recently in [8,9]. Since these methods are well established, we shall state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new multipoint order methods for finding multiple roots.

In [9], Thukral developed a sixth and fifth order of convergence methods, since these methods are well established we state the essential expressions used in each of the methods.

The sixth-order method for finding multiple roots of a nonlinear equation is expressed as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (26)$$

$$z_n = x_n - m \left(\sum_{i=1}^3 i \left(\frac{f(y_n)}{f(x_n)} \right)^{i/m} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (27)$$

$$x_{n+1} = z_n - m \left(\sum_{i=1}^3 i \left(\frac{f(y_n)}{f(x_n)} \right)^{i/m} \right)^2 \times \left(\frac{f(z_n)}{f(x_n)} \right)^{m-1} \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (28)$$

where $n \in \mathbb{N}$, x_0 is the initial value and provided that the denominators of (6) and (7) are not equal to zero.

The second of the method given in [9] is the fifth-order method for finding multiple root of a nonlinear equation is expressed as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (29)$$

$$z_n = x_n - m \left(\sum_{i=1}^3 i \left(\frac{f(y_n)}{f(x_n)} \right)^{i/m} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (30)$$

$$x_{n+1} = z_n - m \left(\frac{f(z_n)}{f(x_n)} \right)^{m-1} \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (31)$$

where $n \in \mathbb{N}$, x_0 is the initial value and provided that the denominators of (20)- (22) are not equal to zero.

In [8] Thukral developed another fifth-order method for finding multiple root of a nonlinear equation is expressed as

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad (32)$$

$$x_{n+1} = y_n - m \left(1 + z^{2/m} \right) \left(\frac{f(y_n)}{f'(y_n)} \right), \quad (33)$$

where $z = f(y_n) f(x_n)^{-1}$, $n \in \mathbb{N}$, x_0 is the initial value and provided that the denominators of (6) and (7) are not equal to zero.

4. Application of the New Multipoint Iterative Methods

The present multipoint methods given by (10) and (24) are employed to solve nonlinear equations with multiple roots and compare with the Thukral methods, (28), (31) and (33). To demonstrate the performance of the new multipoint

methods, we use ten particular nonlinear equations. We determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the new multipoint methods for determining the multiple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

The new multipoint method requires four function evaluations. To determine the efficiency index of the new methods, we shall use the definition 2. Hence, the efficiency

index of the multipoint method given by (10) is $\sqrt[4]{6} \approx 1.565$ whereas the efficiency index of the fifth-order method is given as $\sqrt[4]{5} \approx 1.495$. We can see that the efficiency index of the new sixth-order method is better than the other similar methods.

The test functions and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n for test functions with initial estimate x_0 , are displayed in Table 2. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method. Furthermore, the computational order of convergence (COC) is displayed in Table 3.

Table 1. Test functions and their roots

Functions	m	Roots	Initial Estimate
$f_1(x) = (x^3 + x + 1)^m$	$m = 7$	$\alpha = -0.68232\dots$	$x_0 = -0.9$
$f_2(x) = (xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5)^m$	$m = 4$	$\alpha = -1.207647\dots$	$x_0 = -1.2$
$f_3(x) = ((x-1)^{10} - 1)^m$	$m = 9$	$\alpha = 0$	$x_0 = -0.01$
$f_4(x) = (\exp(x) + x - 20)^m$	$m = 95$	$\alpha = 2.842438\dots$	$x_0 = 3$
$f_5(x) = (\cos(x) + x)^m$	$m = 15$	$\alpha = -0.739085\dots$	$x_0 = -1$
$f_6(x) = (\sin(x)^2 - x^2 + 1)^m$	$m = 500$	$\alpha = 1.404491\dots$	$x_0 = 1.7$
$f_7(x) = (e^{-x^2} - e^{x^2} - x^8 + 10)^m$	$m = 30$	$\alpha = 1.239417\dots$	$x_0 = 1.3$
$f_8(x) = (6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1)^m$	$m = 55$	$\alpha = 0.257530\dots$	$x_0 = -2$
$f_9(x) = (\tan(x) - e^x - 1)^m$	$m = 11$	$\alpha = 1.371045\dots$	$x_0 = 1.4$
$f_{10}(x) = (\ln(x^2 + 3x + 5) - 2x + 7)^m$	$m = 31$	$\alpha = 5.469012\dots$	$x_0 = 6$

Table 2. Comparison of various iterative methods

f_i	(33)	(31)	(24)	(28)	(10)
f_1	0.139e-84	0.239e-81	0.116e-96	0.251e-159	0.814e-187
f_2	0.213e-231	0.629e-80	0.143e-262	0.294e-124	0.172e-501
f_3	0.142e-156	0.287e-155	0.157e-172	0.154e-297	0.169e-324
f_4	0.591e-130	0.649e-128	0.137e-148	0.221e-252	0.468e-287
f_5	0.515e-137	0.495e-136	0.977e-140	0.284e-235	0.103e-240
f_6	0.615e-76	0.106e-73	0.110e-82	0.153e-137	0.711e-153
f_7	0.181e-94	0.128e-91	0.176e-109	0.118e-180	0.853e-213
f_8	0.424e-25	0.983e-23	0.425e-31	0.330e-46	0.241e-62
f_9	0.453e-90	0.220e-86	0.478e-119	0.921e-181	0.626e-248
f_{10}	0.186e-244	0.204e-244	0.899e-246	0.362e-409	0.464e-411

Table 3. COC of various iterative methods

f_i	(33)	(31)	(24)	(28)	(10)
f_1	5.0000	5.0000	5.0000	6.0001	6.0001
f_2	5.0000	5.0002	5.0000	6.0004	6.0000
f_3	4.9999	5.0000	5.0001	6.0000	6.0003
f_4	5.0000	5.0000	4.9999	6.0000	6.0000
f_5	5.0000	5.0000	4.9999	5.9999	6.0000
f_6	5.0000	5.0000	5.0002	6.0001	6.0000
f_7	4.9999	5.0000	5.0000	6.0000	6.0000
f_8	4.9575	4.9645	4.9826	6.0079	5.9989
f_9	5.0000	5.0002	5.0001	6.0001	6.0000
f_{10}	5.0000	5.0000	5.0001	5.9998	6.0002

5. Remarks and Conclusions

In this paper, we have introduced two new multipoint iterative methods for solving nonlinear equations with multiple roots. Convergence analysis proves that the new methods preserve their order of convergence. Simply introducing new parameter in (10) we have achieved sixth-order of convergence. The prime motive of presenting these new methods was to establish a higher order of convergence method than the existing third, fourth and fifth order methods [2-12]. We have examined the effectiveness of the new iterative methods by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that the convergence of the tested multipoint methods of the sixth order is remarkably fast. Furthermore, in most of the test examples, empirically we have found that the best results of the new iterative methods (10) are obtained when $a = 0$. The main purpose of demonstrating the new methods for different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. The major advantages of the new multipoint iterative methods are; first, able to evaluate simple and multiple roots, secondly, simple formula when compared to existing methods containing long expressions of m (see[3]), thirdly, produces better estimate than the similar methods. Finally, we conjecture that new concept introduced in this paper by Thukral may be applied to transform the iterative methods used for finding simple root of nonlinear equations.

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