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New Results for a Caputo Boundary Value Problem

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Abstract In this paper, we study a four point boundary problem for fractional differential equations. We establish new existence and uniqueness results using the Banach contraction principle. Other existence results are generated using the well known Schaefer's fixed point theorem. To illustrate our results, we present some examples for the Banach contraction result. Other examples are also treated to illustrate our second main result.

Keywords Boundary Value Problem, Caputo Derivative, Fixed Point, Riemann-Liouville Integral

1. Introduction

Boundary value problems for fractional differential equations have been shown to be very useful in the study of models of many phenomena in various fields of applied science and engineering, such as electrochemistry, chemistry, visco-elasticity, control, biophysics. For more details, we refer the reader to[3, 4, 9, 11, 14, 15, 16, 20] and references therein. Recently, there has been a significant progress in the study of these equations, (see[7, 8, 19]). More recently, some new theories for the initial boundary value problems of fractional differential equations have been discussed in[1, 2, 12, 13 14]. Moreover, existence and uniqueness of solutions to boundary value problems for fractional differential equations has attracted the attention of many authors[6, 7, 8, 14, 17].

In[17, 18], the existence and uniqueness of solutions were investigated for a nonlinear fractional differential equation with three-point boundary conditions by using a Schauder's fixed point theorem. The existence of solutions for a nonlinear fractional differential equation with four-point boundary conditions was investigated in[5, 10] by using Schauder's fixed point.

Motivated by the problem (1) in[20], this paper deals with the existence and uniqueness of solutions for the following problem:

$$\begin{aligned}
&D^{\alpha}x(t)=f\left(t,x(t),D^{\beta}x(t)\right)=0,t\in[0,1],\\
&x(0)=0,x(1)=0, \qquad (1.1)\\
&\lambda_1 x''(\eta)-\lambda_2 x'''(\eta)=0,\lambda_3 x''(\xi)+\lambda_4 x'''(\xi)=0,
\end{aligned}$$

where $3<\alpha\leq 4,\beta\leq\alpha-1,0<\eta,\xi<1,$ and D^{α} and D^{β} are the Caputo fractional derivatives, $J=[0,1],\lambda_i.i=1,4$ are real constants with $\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_3(\xi-\eta)\neq 0,$ and f is a continuous function on $[0,1]\times\Re^2$. The paper is organized as follows. In section 2, we present some preliminaries and lemmas. In Section 3, we prove the main results of this work. In section 4, we will give some examples to illustrate our results.

2. Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 1: The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a continuous function f on $[0,\infty]$ is defined as:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\alpha>0, \qquad (2.1)$$

$$J^0 f(t)=f(t),$$

where

$$\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$$

Definition 2: The fractional derivative of $f\in C^n([0,\infty])$ in the sense of Caputo is defined as:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \qquad (2.2)$$

$$n-1<\alpha,n\in N^*.$$

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For more details, we refer the reader to[14, 16].

Let us now introduce the following Banach space $X = \{x : x \in C([0,1]), D^{\beta} x \in C([0,1])\}$, endowed with the norm $\|x\|_X = \|x\| + \|D^{\beta} x\|; \|x\| = \sup_{t\in J} |x(t)|, \|D^{\beta} x\| = \sup_{t\in J} |D^{\beta} x(t)|$.

We give the following lemmas[11]:

Lemma 3: For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \Re,\ i = 0,1,2,..,n-1, n = [\alpha]+1$.

Lemma 4: Let $\alpha > 0$. Then

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \tag{2.4}$$

for some $c_i \in \Re,\ i = 0,1,2,...,n-1, n = [\alpha]+1$.

We give also the following result:

Lemma 5: Let $g \in C([0,1])$, the solution of the equation

$$D^{\alpha} x(t) = g(t), t \in J, 3 < \alpha \le 4, \tag{2.5}$$

subject to the boundary conditions

$$\begin{gathered} x(0) = 0, x(1) = 0, \\ \lambda_1 x''(\eta) - \lambda_2 x'''(\eta) = 0, \lambda_3 x''(\xi) + \lambda_4 x'''(\xi) = 0, \end{gathered} \tag{2.6}$$

is given by:

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^{\beta} x(s)) ds \\ &- \frac{t}{6(b+c+a(\xi-\eta))\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), D^{\beta} x(s)) ds \\ &+ \frac{(3c + a(3\xi-1))t - 3(c+a\xi)t^2 + at^3}{6(b+c+a(\xi-\eta))\Gamma(\alpha-2)} \int_0^{\eta} (\eta-s)^{\alpha-3} f(s, x(s), D^{\beta} x(s)) ds \\ &- \frac{(3d + b(3\xi-1))t - 3(d+b\xi)t^2 + bt^3}{6(b+c+a(\xi-\eta))\Gamma(\alpha-3)} \int_0^{\eta} (\eta-s)^{\alpha-4} f(s, x(s), D^{\beta} x(s)) ds \\ &- \frac{(a(3\eta-1) - 3b)t + 3(b-a\eta)t^2 + at^3}{6(b+c+a(\xi-\eta))\Gamma(\alpha-2)} \int_0^{\xi} (\xi-s)^{\alpha-3} f(s, x(s), D^{\beta} x(s)) ds \\ &- \frac{(c(3\eta-1) - 3d)t + 3(d-c\eta)t^2 + ct^3}{6(b+c+a(\xi-\eta))\Gamma(\alpha-3)} \int_0^{\xi} (\xi-s)^{\alpha-4} f(s, x(s), D^{\beta} x(s)) ds, \end{aligned} \tag{2.7}$$

where

$$a = \lambda_1 \lambda_3, b = \lambda_2 \lambda_3, c = \lambda_1 \lambda_4, =, d = \lambda_2 \lambda_4.$$

Proof. For $c_i \in \Re, i = 0,1,2,3,$ and by lemmas 3, 4, the general solution of (2.5) is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.8}$$

Using the boundary conditions (2.6), we obtain $c_0 = 0$,

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds - c_2 - c_3,$$

and

$$
\begin{aligned}
c_2 = & \frac{c+a\xi}{2(b+c+a(\xi-\eta))\Gamma(\alpha-2)}\int_0^\eta (\eta-s)^{\alpha-3} g(s)ds \\
& -\frac{d+b\xi}{2(b+c+a(\xi-\eta))\Gamma(\alpha-3)}\int_0^\eta (\eta-s)^{\alpha-4} g(s)ds \\
& +\frac{b-a\eta}{2(b+c+a(\xi-\eta))\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} g(s)ds \\
& +\frac{d-c\eta}{2(b+c+a(\xi-\eta))\Gamma(\alpha-3)}\int_0^\xi (\xi-s)^{\alpha-4} g(s)ds,
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
c_3 = & -\frac{a}{6(b+c+a(\xi-\eta))\Gamma(\alpha-2)}\int_0^\eta (\eta-s)^{\alpha-3} g(s)ds \\
& +\frac{b}{6(b+c+a(\xi-\eta))\Gamma(\alpha-3)}\int_0^\eta (\eta-s)^{\alpha-4} g(s)ds \\
& +\frac{a}{6(b+c+a(\xi-\eta))\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} g(s)ds \\
& +\frac{c}{6(b+c+a(\xi-\eta))\Gamma(\alpha-3)}\int_0^\xi (\xi-s)^{\alpha-4} g(s)ds.
\end{aligned} \tag{2.10}
$$

Substituting the values of c_1, c_2 and c_3 in (2.8), we obtain the desired relation in Lemma 5.

3. Main Results

We introduce the following quantities

$$
\begin{aligned}
M_1 = & \frac{6|b+c+a(\xi-\eta)|+1}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)} \\
& +\frac{\left(|3c+a(3\xi-1)|+3|c+a\xi|+|a|\right)\eta^{\alpha-2}+\left(|a(3\eta-1)-3b|+3|b-a\eta|+|a|\right)\xi^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-1)} \\
& +\frac{\left(|3d+b(3\xi-1)|+3|d+b\xi|+|b|\right)\eta^{\alpha-3}+\left(|c(3\eta-1)-3d|+3|d-c\eta|+|c|\right)\xi^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-2)}, \\
M_2 = & \frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)\Gamma(2-\beta)} \\
& +\frac{|3c+a(3\xi-1)|\eta^{\alpha-2}+|a(3\eta-1)-3b|\xi^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-1)}+\frac{|c+a\xi|\eta^{\alpha-2}+|b-a\eta|\xi^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-1)} \\
& +\frac{|a|\left(\eta^{\alpha-2}+\xi^{\alpha-2}\right)}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-1)}+\frac{|3d+b(3\xi-1)|\eta^{\alpha-3}+|c(3\eta-1)-3d|\xi^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-2)} \\
& +\frac{|d+b\xi|\eta^{\alpha-3}+|d-c\eta|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-2)}+\frac{|b|\eta^{\alpha-3}+|c|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-2)},
\end{aligned} \tag{3.1}
$$

and we list the following hypotheses:

(H1): The function $f : [0,1]\times\Re^2 \to \Re$ is continuous.

(H2): There exist non negative continuous functions a, b, on J, such that for $t \in J, (x, y), (x_1, y_1) \in \Re^2$, we have

$$\left|f(t,x,y)-f(t,x_1,y_1)\right| \le a(t)\left|x-x_1\right|+b(t)\left|y-y_1\right|, \tag{3.2}$$

where, $\omega=\sup_{t\in J}\left|a(t)\right|$ and $\varpi=\sup_{t\in J}\left|b(t)\right|$.

(H3): There exists $L>0$ such that

$$\left|f(t,x,y)\right| \le L; t\in J, x, y\in\Re. \tag{3.3}$$

Our first result is based on the Banach contraction principle. We have:

Theorem 6: *Assume that the hypothesis* (H2) holds.

If

$$(M_1+M_2)(\omega+\varpi)<1, \tag{3.4}$$

then the problem (1.1) has a unique solution on J.

Proof. Consider the operator $\phi : X \to X$ defined by:

$$\begin{aligned}
\varphi x(t) &:= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s,x(s),D^\beta x(s)\right)ds \\
&-\frac{t}{6\left(b+c+a(\xi-\eta)\right)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\left(s,x(s),D^\beta x(s)\right)ds \\
&+\frac{\left(3c+a(3\xi-1)\right)t-3(c+a\xi)t^2+at^3}{6\left(b+c+a(\xi-\eta)\right)\Gamma(\alpha-2)}\int_0^\eta (\eta-s)^{\alpha-3} f\left(s,x(s),D^\beta x(s)\right)ds \\
&-\frac{\left(3d+b(3\xi-1)\right)t-3(d+b\xi)t^2+bt^3}{6\left(b+c+a(\xi-\eta)\right)\Gamma(\alpha-3)}\int_0^\eta (\eta-s)^{\alpha-4} f\left(s,x(s),D^\beta x(s)\right)ds \\
&-\frac{\left(a(3\eta-1)-3b\right)t+3(b-a\eta)t^2+at^3}{6\left(b+c+a(\xi-\eta)\right)\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} f\left(s,x(s),D^\beta x(s)\right)ds \\
&-\frac{\left(c(3\eta-1)-3d\right)t+3(d-c\eta)t^2+ct^3}{6\left(b+c+a(\xi-\eta)\right)\Gamma(\alpha-3)}\int_0^\xi (\xi-s)^{\alpha-4} f\left(s,x(s),D^\beta x(s)\right)ds.
\end{aligned} \tag{3.5}$$

We shall prove that ϕ is a contraction:

For $x, y \in X$ and $t \in J$, we obtain

$$\begin{aligned}
\left|\varphi x(t)-\varphi y(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds \\
&+\frac{t}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds \\
&+\frac{\left|3c+a(3\xi-1)\right|t+3\left|c+a\xi\right|t^2+\left|a\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^\eta (\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds \\
&+\frac{\left|3d+b(3\xi-1)\right|t+3\left|d+b\xi\right|t^2+\left|b\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^\eta (\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds \\
&+\frac{\left|a(3\eta-1)-3b\right|t+3\left|b-a\eta\right|t^2+\left|a\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds \\
&+\frac{\left|c(3\eta-1)-3d\right|t+3\left|d-c\eta\right|t^2+\left|c\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^\xi (\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds.
\end{aligned} \tag{3.6}$$

Using (H2), we can write:

$$\begin{aligned}\left|\phi x(t)-\phi y(t)\right| \le{}& \frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{\Gamma(\alpha+1)}+\frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)}\\
&+\frac{\left(\left|3c+a(3\xi-1)\right|+3\left|c+a\xi\right|+|a|\right)\eta^{\alpha-2}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\\
&+\frac{\left(\left|3d+b(3\xi-1)\right|+3\left|d+b\xi\right|+|b|\right)\eta^{\alpha-3}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\\
&+\frac{\left(\left|a(3\eta-1)-3b\right|+3\left|b-a\eta\right|+|a|\right)\xi^{\alpha-2}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\\
&+\frac{\left(\left|c(3\eta-1)-3d\right|+3\left|d-c\eta\right|+|c|\right)\xi^{\alpha-3}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}
\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\le\frac{\left(6\left|b+c+a(\xi-\eta)\right|+1\right)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)}\\
&+\frac{\begin{bmatrix}\left(\left|3c+a(3\xi-1)\right|+3\left|c+a\xi\right|+|a|\right)\eta^{\alpha-2}\\ +\left(\left|a(3\eta-1)-3b\right|+3\left|b-a\eta\right|+|a|\right)\xi^{\alpha-2}\end{bmatrix}(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\\
&+\frac{\begin{bmatrix}\left(\left|3d+b(3\xi-1)\right|+3\left|d+b\xi\right|+|b|\right)\eta^{\alpha-3}\\ +\left(\left|c(3\eta-1)-3d\right|+3\left|d-c\eta\right|+|c|\right)\xi^{\alpha-3}\end{bmatrix}(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}.
\end{aligned}$$

Consequently, we obtain,

$$\left|\phi x(t)-\phi y(t)\right|\le M_1(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \tag{3.8}$$

Hence, we have

$$\left\|\varphi(x)-\varphi(y)\right\|\le M_1(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \tag{3.9}$$

On the other hand,

$$\begin{aligned}\left|D^{\beta}\varphi x(t)-D^{\beta}\varphi y(t)\right|\le{}&\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds\\
&+\frac{t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-2)}\begin{bmatrix}\dfrac{\left|3c+a(3\xi-1)\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\ +\dfrac{\left|c+a\xi\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\ +\dfrac{|a|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{bmatrix}\int_0^{\eta}(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\frac{|3d+b(3\xi-1)|t^{1-\beta}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\frac{|d+b\xi|t^{2-\beta}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\frac{|b|t^{3-\beta}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\int_0^{\eta}(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds \quad (3.10)$$

$$+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\frac{|a(3\eta-1)-3b|t^{1-\beta}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\frac{|b-a\eta|t^{2-\beta}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\frac{|a|t^{3-\beta}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds$$

$$+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\frac{|c(3\eta-1)-3d|t^{1-\beta}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\frac{|d-c\eta|t^{2-\beta}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\frac{|c|t^{3-\beta}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds.$$

By $(H2)$, we obtain

$$\left|D^{\beta}\varphi x(t)-D^{\beta}\varphi y(t)\right|\leq\frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{\Gamma(\alpha-\beta+1)}+\frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)\Gamma(2-\beta)}$$

$$+\frac{1}{\Gamma(\alpha-1)}\left[\begin{array}{l}\frac{|3c+a(3\xi-1)|\eta^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\frac{|c+a\xi|\eta^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\frac{|a|\eta^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|$$

$$+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\frac{|3d+b(3\xi-1)|\eta^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\frac{|d+b\xi|\eta^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\frac{|b|\eta^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|$$

$$+\frac{1}{\Gamma(\alpha-1)}\left[\begin{array}{l}\dfrac{|a(3\eta-1)-3b|\xi^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\dfrac{|b-a\eta|\xi^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\dfrac{|a|\xi^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|$$

$$+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{|c(3\eta-1)-3d|\xi^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)}\\+\dfrac{|d-c\eta|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)}\\+\dfrac{|c|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)}\end{array}\right]\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|. \tag{3.11}$$

Hence,

$$\left|D^\beta\varphi x(t)-D^\beta\varphi y(t)\right|\le\left[\frac{\omega+\varpi}{\Gamma(\alpha-\beta+1)}+\frac{\omega+\varpi}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right)$$

$$+\omega+\varpi\left[\begin{array}{l}\dfrac{|3c+a(3\xi-1)|\eta^{\alpha-2}+|a(3\eta-1)-3b|\xi^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-1)}\\+\dfrac{|c+a\xi|\eta^{\alpha-2}+|b-a\eta|\xi^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-1)}\\+\dfrac{|a|\left(\eta^{\alpha-2}+\xi^{\alpha-2}\right)}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-1)}\end{array}\right]\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right) \tag{3.12}$$

$$+\omega+\varpi\left[\begin{array}{l}\dfrac{|3d+b(3\xi-1)|\eta^{\alpha-3}+|c(3\eta-1)-3d|\xi^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-2)}\\+\dfrac{|d+b\xi|\eta^{\alpha-3}+|d-c\eta|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-2)}\\+\dfrac{|b|\eta^{\alpha-3}+|c|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-2)}\end{array}\right]\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right).$$

Therefore,

$$\left|D^\beta\phi x(t)-D^\beta\phi y(t)\right|\le M_2(\omega+\varpi)\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right), \tag{3.13}$$

which implies,

$$\left\|D^\beta\phi(x)-D^\beta\phi(y)\right\|\le M_2(\omega+\varpi)\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right). \tag{3.14}$$

It follows from (3.9) and (3.14) that

$$\|\phi(x)-\phi(y)\|_X\le(M_1+M_2)(\omega+\varpi)\left(\|x-y\|+\left\|D^\beta x-D^\beta y\right\|\right). \tag{3.15}$$

Thanks to (3.4), we deduce that φ is a contraction. As a consequence of Banach contraction principle, the problem (1.1) has a unique solution on J.

The second result is based on Scheafer's fixed point theorem.

Theorem7: *Suppose that (H1) and (H3) hold.*

Then, the problem (1.1) has at least one solution on J.

Proof. We use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on X. The proof will be given in following steps:

Step1: ϕ is continuous on X: In view of the continuity of f, we conclude that the operator ϕ is continuous.

Step2: The operator ϕ maps bounded sets into bounded sets in X: For $\sigma > 0$, we take $x \in B_\sigma = \{x \in X; \|x\|_X \leq \sigma\}$.

For $x \in B_\sigma$, and $t \in J$, we can write:

$$\begin{aligned}
|\varphi x(t)| \leq{}& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{t}{6|b+c+a(\xi-\eta)|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{|3c+a(3\xi-1)|t + 3|c+a\xi|t^2 + |a|t^3}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-2)} \int_0^\eta (\eta-s)^{\alpha-3} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{|3d+b(3\xi-1)|t + 3|d+b\xi|t^2 + |b|t^3}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-3)} \int_0^\eta (\eta-s)^{\alpha-4} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{|a(3\eta-1)-3b|t + 3|b-a\eta|t^2 + |a|t^3}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-2)} \int_0^\xi (\xi-s)^{\alpha-3} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{|c(3\eta-1)-3d|t + 3|d-c\eta|t^2 + |c|t^3}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-3)} \int_0^\xi (\xi-s)^{\alpha-4} \left| f\left(s, x(s), D^\beta x(s)\right)\right| ds.
\end{aligned} \tag{3.16}$$

Using (H3), we can write

$$\begin{aligned}
|\varphi x(t)| \leq{}& \frac{L\left(6|b+c+a(\xi-\eta)|+1\right)}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)} \\
&+ \frac{L\begin{bmatrix} \left(|3c+a(3\xi-1)|+3|c+a\xi|+|a|\right)\eta^{\alpha-2} \\ +\left(|a(3\eta-1)-3b|+3|b-a\eta|+|a|\right)\xi^{\alpha-2} \end{bmatrix}}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-1)} \\
&+ \frac{L\begin{bmatrix} \left(|3d+b(3\xi-1)|+3|d+b\xi|+|b|\right)\eta^{\alpha-3} \\ +\left(|c(3\eta-1)-3d|+3|d-c\eta|+|c|\right)\xi^{\alpha-3} \end{bmatrix}}{6|b+c+a(\xi-\eta)|\Gamma(\alpha-2)}.
\end{aligned} \tag{3.17}$$

Thus,

$$|\phi x(t)| \leq LM_1, t \in J, \tag{3.18}$$

which implies that

$$\|\phi(x)\| \le LM_1, \tag{3.19}$$

and

$$\begin{aligned}
\left|D^{\beta}\varphi x(t)\right| \le{}& \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{\left|3c+a(3\xi-1)\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)} \\ +\dfrac{\left|c+a\xi\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)} \\ +\dfrac{\left|a\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^{\eta}(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\dfrac{\left|3d+b(3\xi-1)\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)} \\ +\dfrac{\left|d+b\xi\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)} \\ +\dfrac{\left|b\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^{\eta}(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{\left|a(3\eta-1)-3b\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)} \\ +\dfrac{\left|b-a\eta\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)} \\ +\dfrac{\left|a\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\dfrac{\left|c(3\eta-1)-3d\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)} \\ +\dfrac{\left|d-c\eta\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)} \\ +\dfrac{\left|c\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.
\end{aligned}$$

By (H3) , yields

$$\left|D^{\beta}\varphi x(t)\right| \le L\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)\Gamma(2-\beta)}\right]$$
$$+L\left[\begin{array}{l}\frac{\left|3c+a(3\xi-1)\right|\eta^{\alpha-2}+\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)\Gamma(\alpha-1)}+\frac{\left|c+a\xi\right|\eta^{\alpha-2}+\left|b-a\eta\right|\xi^{\alpha-2}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)\Gamma(\alpha-1)}\\ +\frac{\left|a\right|\left(\eta^{\alpha-2}+\xi^{\alpha-2}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)\Gamma(\alpha-1)}\end{array}\right] \quad (3.21)$$
$$+L\left[\begin{array}{l}\frac{\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}+\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)\Gamma(\alpha-2)}+\frac{\left|d+b\xi\right|\eta^{\alpha-3}+\left|d-c\eta\right|\xi^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)\Gamma(\alpha-2)}\\ +\frac{\left|b\right|\eta^{\alpha-3}+\left|c\right|\xi^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)\Gamma(\alpha-2)}\end{array}\right].$$

Hence,

$$\left|D^{\beta}\phi x(t)\right| \le LM_2, t\in J. \quad (3.22)$$

And consequently,

$$\left\|D^{\beta}\phi(x)\right\| \le LM_2. \quad (3.23)$$

Thanks to (3.19) and (3.23), we obtain

$$\left\|\phi(x)\right\|_X \le L(M_1+M_2). \quad (3.24)$$

Therefore,

$$\left\|\phi(x)\right\|_X < \infty.$$

Step3: The operator ϕ is equicontinuous on *X:*

Let us take $x\in B_{\sigma}, t_1, t_2\in J, t_1<t_2$. We have:

$$\begin{aligned}\left|\varphi x(t_2)-\varphi x(t_1)\right| \le & \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{t_1-t_2}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{\left|3c+a(3\xi-1)\right|(t_2-t_1)+3\left|c+a\xi\right|\left(t_2^2-t_1^2\right)+\left|a\right|\left(t_2^3-t_1^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^{\eta}(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \quad (3.25)\\ &+\frac{\left|3d+b(3\xi-1)\right|(t_1-t_2)+3\left|d+b\xi\right|\left(t_1^2-t_2^2\right)+\left|b\right|\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^{\eta}(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{\left|a(3\eta-1)-3b\right|(t_1-t_2)+3\left|b-a\eta\right|\left(t_1^2-t_2^2\right)+\left|a\right|\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{\left|c(3\eta-1)-3d\right|(t_1-t_2)+3\left|d-c\eta\right|\left(t_1^2-t_2^2\right)+\left|c\right|\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^{\xi}(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.\end{aligned}$$

Thanks to (H3), we can write

$$\left|\varphi x(t_2)-\varphi x(t_1)\right|\le \frac{L\left(t_2^{\alpha}-t_1^{\alpha}\right)}{\Gamma(\alpha+1)}+\frac{L(t_1-t_2)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)}$$
$$+\frac{L\left|3c+a(3\xi-1)\right|\eta^{\alpha-2}(t_2-t_1)+3L\left|c+a\xi\right|\eta^{\alpha-2}\left(t_2^2-t_1^2\right)+L|a|\eta^{\alpha-2}\left(t_2^3-t_1^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}$$
$$+\frac{L\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}(t_1-t_2)+3L\left|d+b\xi\right|\eta^{\alpha-3}\left(t_1^2-t_2^2\right)+L|b|\eta^{\alpha-3}\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}$$
$$+\frac{L\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}(t_1-t_2)+3L\left|b-a\eta\right|\xi^{\alpha-2}\left(t_1^2-t_2^2\right)+L|a|\xi^{\alpha-2}\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}$$
$$+\frac{L\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}(t_1-t_2)+3L\left|d-c\eta\right|\xi^{\alpha-3}\left(t_1^2-t_2^2\right)+L|c|\xi^{\alpha-3}\left(t_1^3-t_2^3\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}. \tag{3.26}$$

Then,

$$\begin{aligned}\left|\phi x(t_2)-\phi x(t_1)\right|\le{}&\frac{L}{6\left|b+c+a(\xi-\eta)\right|}\left[\begin{array}{l}\dfrac{1}{\Gamma(\alpha+1)}+\dfrac{\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\\+\dfrac{\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}+\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}\end{array}\right](t_1-t_2)\\&\frac{L}{2\left|b+c+a(\xi-\eta)\right|}\left[\frac{\left|d+b\xi\right|\eta^{\alpha-3}+\left|d-c\eta\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|b-a\eta\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^2-t_2^2\right)\\&+\frac{L}{6\left|b+c+a(\xi-\eta)\right|}\left[\frac{|b|\eta^{\alpha-3}+|c|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{|a|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^3-t_2^3\right)\\&\frac{L\left|3c+a(3\xi-1)\right|\eta^{\alpha-2}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}(t_2-t_1)+\frac{L\left|c+a\xi\right|\eta^{\alpha-2}}{2\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\left(t_2^2-t_1^2\right)\\&\frac{L|a|\eta^{\alpha-2}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{L}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right),\end{aligned} \tag{3.27}$$

we have also,

$$\left|D^{\beta}\varphi x(t_2)-D^{\beta}\varphi x(t_1)\right|\le\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right)\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds$$
$$+\frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds$$
$$+\frac{\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{\left|3c+a(3\xi-1)\right|\left(t_2^{1-\beta}-t_1^{1-\beta}\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\+\dfrac{\left|c+a\xi\right|\left(t_2^{2-\beta}-t_1^{2-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\+\dfrac{\left|a\right|\left(t_2^{3-\beta}-t_1^{3-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\eta(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\dfrac{\left|3d+b(3\xi-1)\right|\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\+\dfrac{\left|d+b\xi\right|\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\+\dfrac{\left|b\right|\left(t_1^{3-\beta}-t_2^{3-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\eta(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{\left|a(3\eta-1)-3b\right|\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\+\dfrac{\left|b-a\eta\right|\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\+\dfrac{\left|a\right|\left(t_1^{3-\beta}-t_2^{3-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\dfrac{\left|c(3\eta-1)-3d\right|\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\+\dfrac{\left|d-c\eta\right|\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\+\dfrac{\left|c\right|\left(t_1^{3-\beta}-t_2^{3-\beta}\right)}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\xi(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds.
\end{aligned}
\tag{3.28}
$$

By (H3), we obtain:

$$\begin{aligned}
&\left|D^{\beta}\varphi x(t_2)-D^{\beta}\varphi x(t_1)\right| \\
&\le \frac{L}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\left[\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}+\frac{\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)} \\ +\frac{\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}+\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}\end{array}\right]\left(t_1^{1-\beta}-t_2^{1-\beta}\right) \\
&+\frac{L}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\left[\frac{\left|c+a\xi\right|\eta^{\alpha-3}+\left|d-c\eta\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|b-a\eta\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^{2-\beta}-t_2^{2-\beta}\right) \\
&+\frac{L}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\left[\frac{\left|b\right|\eta^{\alpha-3}+\left|c\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|a\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^{3-\beta}-t_2^{3-\beta}\right) \\
&+\frac{L\left|3c+a(3\xi-1)\right|\eta^{\alpha-3}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(2-\beta)}\left(t_2^{1-\beta}-t_1^{1-\beta}\right) \\
&+\frac{L\left|c+a\xi\right|\eta^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(3-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right) \\
&+\frac{L\left|a\right|\eta^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(4-\beta)}\left(t_2^{3-\beta}-t_1^{3-\beta}\right)+\frac{L}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right).
\end{aligned} \tag{3.29}$$

Hence, from (3.27) and (3.29), we get

$$\begin{aligned}
\left\|\phi x(t_2)-\phi x(t_1)\right\|_X \le{}& \frac{L}{6\left|b+c+a(\xi-\eta)\right|}\left[\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}+\frac{\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)} \\ +\frac{\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}+\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}\end{array}\right](t_1-t_2) \\
&+\frac{L}{2\left|b+c+a(\xi-\eta)\right|}\left[\frac{\left|d+b\xi\right|\eta^{\alpha-3}+\left|d-c\eta\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|b-a\eta\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^2-t_2^2\right) \\
&+\frac{L}{6\left|b+c+a(\xi-\eta)\right|}\left[\frac{\left|b\right|\eta^{\alpha-3}+\left|c\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|a\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^3-t_2^3\right) \\
&+\frac{L\left|3c+a(3\xi-1)\right|\eta^{\alpha-2}}{\Gamma(\alpha-1)6\left|b+c+a(\xi-\eta)\right|}(t_2-t_1)+\frac{L\left|c+a\xi\right|\eta^{\alpha-2}}{2\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\left(t_2^2-t_1^2\right) \\
&+\frac{L\left|a\right|\eta^{\alpha-2}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{L}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{L}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\left[\begin{array}{l}\frac{1}{\Gamma(\alpha+1)}+\frac{\left|a(3\eta-1)-3b\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\\+\frac{\left|3d+b(3\xi-1)\right|\eta^{\alpha-3}+\left|c(3\eta-1)-3d\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}\end{array}\right]\left(t_1^{1-\beta}-t_2^{1-\beta}\right)\\
&+\frac{L}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\left[\frac{\left|c+a\xi\right|\eta^{\alpha-3}+\left|d-c\eta\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|b-a\eta\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^{2-\beta}-t_2^{2-\beta}\right)\\
&+\frac{L}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\left[\frac{\left|b\right|\eta^{\alpha-3}+\left|c\right|\xi^{\alpha-3}}{\Gamma(\alpha-2)}+\frac{\left|a\right|\xi^{\alpha-2}}{\Gamma(\alpha-1)}\right]\left(t_1^{3-\beta}-t_2^{3-\beta}\right)\\
&+\frac{L\left|3c+a(3\xi-1)\right|\eta^{\alpha-3}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(2-\beta)}\left(t_2^{1-\beta}-t_1^{1-\beta}\right)+\frac{L\left|c+a\xi\right|\eta^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(3-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right)\\
&+\frac{L\left|a\right|\eta^{\alpha-3}}{\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)\Gamma(4-\beta)}\left(t_2^{3-\beta}-t_1^{3-\beta}\right)+\frac{L}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right),
\end{aligned}
\tag{3.30}
$$

which implies $\left\|\phi x(t_2)-\phi x(t_1)\right\|_X \to 0$ as $t_2 \to t_1$. By Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

Step4: We show that the set Ω defined by:

$$\Omega=\{x\in X, x=\mu\phi(x), 0<\mu<1\} \tag{3.31}$$

is bounded:

Let $x\in\Omega$, then $x=\mu\phi(x)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$
\begin{aligned}
\frac{1}{\mu}\left|x(t)\right|\le&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{t}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{\left|3c+a(3\xi-1)\right|t+3\left|c+a\xi\right|t^2+\left|a\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^\eta(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{\left|3d+b(3\xi-1)\right|t+3\left|d+b\xi\right|t^2+\left|b\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^\eta(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{\left|a(3\eta-1)-3b\right|t+3\left|b-a\eta\right|t^2+\left|a\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\
&+\frac{\left|c(3\eta-1)-3d\right|t+3\left|d-c\eta\right|t^2+\left|c\right|t^3}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-3)}\int_0^\xi(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds.
\end{aligned}
\tag{3.32}
$$

Thanks to (H3), we can write

$$\frac{1}{\mu}\left|x(t)\right| \le \frac{L\left(6\left|b+c+a(\xi-\eta)\right|+1\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)}$$
$$\frac{L\left[\left(\left|3c+a(3\xi-1)\right|+3\left|c+a\xi\right|+\left|a\right|\right)\eta^{\alpha-2}+\left(\left|a(3\eta-1)-3b\right|+3\left|b-a\eta\right|+\left|a\right|\right)\xi^{\alpha-2}\right]}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)} \tag{3.33}$$
$$+\frac{L\left[\left(\left|3d+b(3\xi-1)\right|+3\left|d+b\xi\right|+\left|b\right|\right)\eta^{\alpha-3}+\left(\left|c(3\eta-1)-3d\right|+3\left|d-c\eta\right|+\left|c\right|\right)\xi^{\alpha-3}\right]}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}.$$

Therefore,

$$\begin{aligned}\left|x(t)\right| \le{}& \mu\frac{L\left(6\left|b+c+a(\xi-\eta)\right|+1\right)}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha+1)}\\ &+\mu\frac{L\left[\left(\left|3c+a(3\xi-1)\right|+3\left|c+a\xi\right|+\left|a\right|\right)\eta^{\alpha-2}+\left(\left|a(3\eta-1)-3b\right|+3\left|b-a\eta\right|+\left|a\right|\right)\xi^{\alpha-2}\right]}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-1)}\\ &+\mu\frac{L\left[\left(\left|3d+b(3\xi-1)\right|+3\left|d+b\xi\right|+\left|b\right|\right)\eta^{\alpha-3}+\left(\left|c(3\eta-1)-3d\right|+3\left|d-c\eta\right|+\left|c\right|\right)\xi^{\alpha-3}\right]}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha-2)}.\end{aligned} \tag{3.34}$$

Thus,

$$\left|x(t)\right| \le \mu M_1, t\in J, \tag{3.35}$$

which implies that,

$$\left\|x\right\| \le \mu L M_1. \tag{3.36}$$

On the other hand,

$$\begin{aligned}\frac{1}{\mu}\left|D^\beta \varphi x(t)\right| \le{}& \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\ &+\frac{t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\ &+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\dfrac{\left|3c+a(3\xi-1)\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\ +\dfrac{\left|c+a\xi\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\ +\dfrac{\left|a\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\eta(\eta-s)^{\alpha-3}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\\ &+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\dfrac{\left|3d+b(3\xi-1)\right|t^{1-\beta}}{6\left|b+c+a(\xi-\eta)\right|\Gamma(2-\beta)}\\ +\dfrac{\left|d+b\xi\right|t^{2-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(3-\beta)}\\ +\dfrac{\left|b\right|t^{3-\beta}}{\left|b+c+a(\xi-\eta)\right|\Gamma(4-\beta)}\end{array}\right]\int_0^\eta(\eta-s)^{\alpha-4}\left|f\left(s,x(s),D^\beta x(s)\right)\right|ds\end{aligned} \tag{3.37}$$

$$+\frac{1}{\Gamma(\alpha-2)}\left[\begin{array}{l}\frac{|a(3\eta-1)-3b|\,t^{1-\beta}}{6|b+c+a(\xi-\eta)|\,\Gamma(2-\beta)}\\+\frac{|b-a\eta|\,t^{2-\beta}}{|b+c+a(\xi-\eta)|\,\Gamma(3-\beta)}\\+\frac{|a|\,t^{3-\beta}}{|b+c+a(\xi-\eta)|\,\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds$$

$$+\frac{1}{\Gamma(\alpha-3)}\left[\begin{array}{l}\frac{|c(3\eta-1)-3d|\,t^{1-\beta}}{6|b+c+a(\xi-\eta)|\,\Gamma(2-\beta)}\\+\frac{|d-c\eta|\,t^{2-\beta}}{|b+c+a(\xi-\eta)|\,\Gamma(3-\beta)}\\+\frac{|c|\,t^{3-\beta}}{|b+c+a(\xi-\eta)|\,\Gamma(4-\beta)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-4}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.$$

By (H3), we have,

$$\begin{aligned}\frac{1}{\mu}\left|D^{\beta}\varphi x(t)\right| &\le L\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1}{6|b+c+a(\xi-\eta)|\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\\&+L\left[\begin{array}{l}\frac{|3c+a(3\xi-1)|\eta^{\alpha-2}+|a(3\eta-1)-3b|\xi^{\alpha-2}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-1)}+\frac{|c+a\xi|\eta^{\alpha-2}+|b-a\eta|\xi^{\alpha-2}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-1)}\\+\frac{|a|\left(\eta^{\alpha-2}+\xi^{\alpha-2}\right)}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-1)}\end{array}\right]\\&+L\left[\begin{array}{l}\frac{|3d+b(3\xi-1)|\eta^{\alpha-3}+|c(3\eta-1)-3d|\xi^{\alpha-3}}{6|b+c+a(\xi-\eta)|\Gamma(2-\beta)\Gamma(\alpha-2)}+\frac{|d+b\xi|\eta^{\alpha-3}+|d-c\eta|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(3-\beta)\Gamma(\alpha-2)}\\+\frac{|b|\eta^{\alpha-3}+|c|\xi^{\alpha-3}}{|b+c+a(\xi-\eta)|\Gamma(4-\beta)\Gamma(\alpha-2)}\end{array}\right].\end{aligned}\tag{3.38}$$

Therefore,

$$\left|D^{\beta}\phi x(t)\right|\le \mu L M_2,\ t\in J.\tag{3.39}$$

Thus,

$$\left\|D^{\beta}x\right\|\le \mu L M_2.\tag{3.40}$$

From (3.36) and (3.40), we get

$$\|x\|_X\le \mu L(M_1+M_2).\tag{3.41}$$

Hence,

$$\|\phi(x)\|_X<\infty.$$

This shows that Ω is bounded.
As consequence of Schaefer's fixed point theorem, the problem (1.1) has at least one solution on *J*.

4. Examples

Examlpe 1: Consider the following problem:

$$D^{\frac{7}{2}}x(t)=\frac{2\pi e^{-\pi t}|x(t)|}{\left(16\sqrt{\pi}+e^{-\pi t}\right)\left(\pi+|x(t)|\right)}+\frac{\sqrt{\pi}\cos(\pi t)\left|D^{\frac{5}{2}}x(t)\right|}{16(t+2)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)},t\in[0,1], \quad (4.1)$$

$$x(0)=0,x(1)=0,$$

$$\frac{3}{4}x''\left(\frac{3}{5}\right)-\frac{2}{5}x'''\left(\frac{3}{5}\right)=0,\frac{4}{5}x''\left(\frac{2}{3}\right)+\frac{1}{3}x'''\left(\frac{2}{3}\right)=0.$$

For this example, we have

$$f(t,x,y)=\frac{2\pi e^{-\pi t}x}{\left(16\sqrt{\pi}+e^{-\pi t}\right)(\pi+x)}+\frac{\sqrt{\pi}\cos(\pi t)y}{16\pi(t+2)^2(1+y)},t\in J:=[0,1],x,y\in[0,\infty).$$

Let $x,y,x_1,y_1\in[0,\infty)$ and $t\in J$. Then we can state that:

$$\begin{aligned}\left|f(t,x,y)-f(t,x_1,y_1)\right|&=\frac{2\pi e^{-\pi t}}{\left(16\sqrt{\pi}+e^{-\pi t}\right)}\left|\frac{x}{\pi+x}-\frac{x_1}{\pi+x_1}\right|\\&\quad+\frac{\sqrt{\pi}\left|\cos(\pi t)\right|}{16\pi(t+2)^2}\left|\frac{y}{1+y}-\frac{y_1}{1+y_1}\right|\\&\le\frac{2\pi e^{-2\pi t}|x-x_1|}{\left(16\sqrt{\pi}+e^{-\pi t}\right)(\pi+x)(\pi+x_1)}+\frac{\sqrt{\pi}\left|\cos(\pi t)\right||y-y_1|}{16\pi(t+2)^2(1+y)(1+y_1)}\\&\le\frac{2\pi e^{-2\pi t}}{\left(16\sqrt{\pi}+e^{-\pi t}\right)}|x-x_1|+\frac{\sqrt{\pi}}{16\pi(t+2)^2}|y-y_1|.\end{aligned}$$

So we can take

$$a(t)=\frac{2\pi e^{-2\pi t}}{\left(16\sqrt{\pi}+e^{-\pi t}\right)},b(t)=\frac{\sqrt{\pi}}{16\pi(t+2)^2}.$$

Therefore,

$$\omega=\sup_{t\in[0,1]}a(t)=\frac{2\pi}{\left(16\sqrt{\pi}+1\right)},\varpi=\sup_{t\in[0,1]}b(t)=\frac{\sqrt{\pi}}{64\pi},$$

and then,

$$M_1=1{,}07714,M_2=1{,}94137.$$

We have also

$$(M_1+M_2)(\omega+\varpi)=0{,}67242<1.$$

Hence by Theorem 6, the problem (4.1) has a unique solution on $[0,1]$.

Example 2: Consider the following problem:

$$D^{\frac{11}{3}}x(t)=\frac{1}{15\left(t^2+1\right)^2}\left(\sin|x(t)|+\frac{\left(t+t^2\right)\sin\left(\left|D^{\frac{7}{3}}x(t)\right|\right)}{3\sqrt{\pi}}\right)+\cosh\left(1+t^2\right),t\in[0,1],$$

$$x(0)=0, x(1)=0,$$
$$\frac{1}{4}x''\left(\frac{1}{2}\right)-\frac{2}{3}x'''\left(\frac{1}{2}\right)=0, \frac{1}{5}x''\left(\frac{3}{4}\right)+\frac{5}{6}x'''\left(\frac{3}{4}\right)=0. \tag{4.2}$$

Set

$$f(t,x,y)=\frac{1}{15(t^2+1)^2}\left(\sin|x|+\frac{(t+t^2)\sin\left(\left|D^{\frac{7}{3}}y\right|\right)}{3\sqrt{\pi}}\right)+\cosh(1+t^2), x, y\in\Re.$$

For $t\in[0,1]$ and $x,y,x_1,y_1\in\Re$, we have:

$$|f(t,x,y)-f(t,x_1,y_1)|\le\frac{1}{15(t^2+1)^2}|x-x_1|+\frac{(t+t^2)}{45\sqrt{\pi}(t^2+1)^2}|y-y_1|.$$

So, we have

$$a(t)=\frac{1}{15(t^2+1)^2}, b(t)=\frac{1}{45\sqrt{\pi}(t^2+1)^2}.$$

Hence,

$$\omega=\sup_{t\in[0,1]}a(t)=\frac{1}{15}, \varpi=\sup_{t\in[0,1]}b(t)=\frac{1}{45\sqrt{\pi}},$$
$$M_1=3,02688, M_2=1,65165,$$

and

$$(M_1+M_2)(\omega+\varpi)=0,370539<1.$$

By Theorem6, the problem (4.2) has a unique solution on $[0,1]$.

Example 3: Our third example is the following:

$$D^{\frac{11}{3}}x(t)=\frac{1}{32(t^2+1)}\left(\sin x(t)+\frac{|\cos(\pi t)|}{5\pi}\sin\left(2\pi D^{\frac{9}{4}}x(t)\right)\right)+\arctan(1+t^2), t\in[0,1],$$
$$x(0)=0, x(1)=0,$$
$$\frac{3}{7}x''\left(\frac{1}{4}\right)-\frac{4}{9}x'''\left(\frac{1}{4}\right)=0, \frac{5}{8}x''\left(\frac{2}{5}\right)+\frac{7}{8}x'''\left(\frac{2}{5}\right)=0. \tag{4.3}$$

For this example, we have:

$$f(t,x,y)=\frac{1}{32(t^2+1)}\left(\sin x+\frac{|\cos(\pi t)|}{5\pi}\sin(2\pi y)\right)+\arctan(1+t^2), t\in[0,1], x,y\in\Re.$$

It is clear that *f* is continuous, and there exists $L>0$, such that $|f(t,x,y)|\le L.$

By theorem7, we can state that the problem (4.3) has at least one solution on $[0,1]$.

Example 4: We give also the following example:

$$D^{\frac{7}{2}}x(t)=\frac{|x(t)|}{7(t+3)^2(3+|x(t)|)}+\frac{t\left|D^{\frac{11}{4}}x(t)\right|}{2\pi(t^2+3\sqrt{\pi})\left(5+\left|D^{\frac{11}{4}}x(t)\right|\right)}+\ln(1+t^2), t\in[0,1],$$
$$x(0)=0, x(1)=0,$$
$$\frac{4}{5}x''\left(\frac{3}{7}\right)-\frac{7}{9}x'''\left(\frac{3}{7}\right)=0, \frac{5}{8}x''\left(\frac{5}{6}\right)+\frac{3}{4}x'''\left(\frac{5}{6}\right)=0. \tag{4.4}$$

It is clear that

$$f(t,x,y)=\frac{|x|}{7(t+3)^2(3+|x|)}+\frac{t|y|}{2\pi(t^2+3\sqrt{\pi})(5+|y|)}+\ln(1+t^2),t\in[0,1],x,y\in\Re.$$

The conditions (H1) and (H3) of Theorem7 are satisfied. Therefore the problem (4.4) has at least one solution on $[0,1]$.

5. Conclusions

In this paper, we have studied a four point boundary problem for fractional differential equations in the sense of Caputo. By using Banach contraction principle, we have established new sufficient conditions for the existence and uniqueness of solutions for the problem (1.1). Other existence results are generated using the well known Schaefer's fixed point theorem. Furthermore, to illustrate our main results, we have treated two examples related to the Banach contraction result. We have also studied two other examples for our second main result.

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