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New Ordinate-Abscissa Based Iterative Schemes to Solve Nonlinear Algebraic Equations

Rajesh C. Shah[1,*], Rajiv B. Shah[2]

[1]Department of Applied Mathematics, Faculty of Technology and Engineering, The M. S. University of Baroda, Vadodara, 390 001, Gujarat State, India
[2]Department of Applied Mathematics, Polytechnic, The M. S. University of Baroda, Vadodara, 390 002, Gujarat State, India

Abstract The aim of the present paper is to propose new iterative schemes for finding a root of a nonlinear algebraic equation $f(x) = 0$ for a given initial guess value x_0. The iterative schemes can be obtained using Taylor's series expansion for $f(x_0 \pm h)$. From the convergence analysis, the various orders of convergence are obtained like linear, order 2 and order 3/2. Some examples are also discussed and compared with Newton's method. Some iterative scheme also works even if $f'(x)=0$, which is the limitation of the Newton's method.

Keywords Iterative Method, Ordinate – Abscissa Method, Algebraic Equations

1. Introduction

Finding a root of an algebraic equation is always curiosity for many researcher in their science and engineering problems. Among the various techniques, Newton's method [1–4] is the most popular and having quadratic convergence. Many researchers[5–15] have tried to develop new iterative scheme in their analysis for faster convergence and claim their advantage over Newton's method.

In this paper, various new iterative schemes (Ordinate – Abscissa based Schemes) are proposed. The various possibility of convergences like linear convergence, convergence of order 3/2, and convergence of order 2 are obtained.

Moreover, some proposed scheme also works even if

$$f'(x) = 0,$$

which is the limitation of the Newton's method.

2. The Basic Idea of Various New Iterative Schemes

The geometrical view of the method is as shown in figure 1.

Consider the following nonlinear algebraic equation

$$f(x) = 0, \tag{1}$$

for which one or more real roots to be found.

Considering x_0 as an initial guess value and x^* as exact root, then from the figure 1

$$PQ^2 = PR^2 + RQ^2,$$

which implies

$$\{f(x_1)\}^2 = \frac{[f(x_0)]^2}{m^2}; m \geq 2, m \in \mathrm{N}, \tag{2}$$

where m indicates number of divisions of ordinate (in our case $f(x_0)$).

Case I When $x_0 > x^*$

Let

$$x_1 = x_0 - h, \tag{3}$$

be the first approximation to the required root of equation (1).

Then equation (2) becomes

$$\{f(x_0 - h)\}^2 = \frac{[f(x_0)]^2}{m^2}. \tag{4}$$

(A) Using Taylor's series expansion and neglecting terms containing $O(h^2)$ and higher as

$$f(x_0 - h) = f(x_0) - hf'(x_0) + O(h^2), \tag{5}$$

equation (4) becomes

$$[m^2 f'^2(x_0)]h^2 + [-2m^2 f(x_0) f'(x_0)]h + (m^2 - 1) f^2(x_0) = 0. \tag{6}$$

Considering

$$a = m^2 f'^2(x_0),$$
$$b = -2m^2 f(x_0) f'(x_0),$$
$$c = (m^2 - 1) f^2(x_0),$$

equation (6) becomes $ah^2 + bh + c = 0$,

which on solving for h gives

$$h = \left(1 \pm \frac{1}{m}\right) \frac{f(x_0)}{f'(x_0)}. \tag{7}$$

* Corresponding author:
dr_rcshah@yahoo.com (Rajesh C. Shah)
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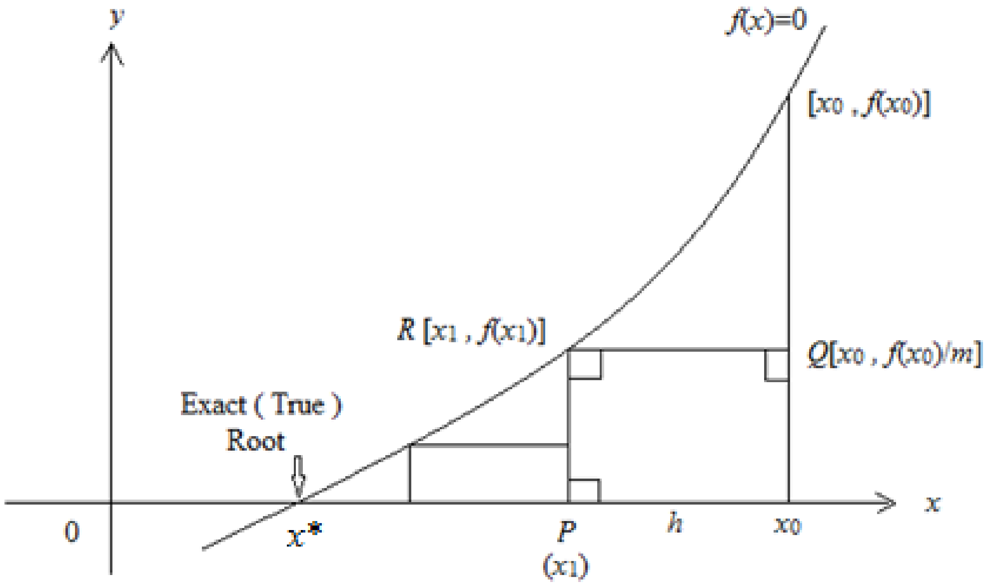

Figure 1. Ordinate - Abscissa method

Therefore, from equation (3) the first approximation can be obtained as

$$x_1 = x_0 - \left(1 \pm \frac{1}{m}\right)\frac{f(x_0)}{f'(x_0)}. \tag{8}$$

The general formula for successive approximation is, therefore, given by

$$x_{n+1} = x_n - h\,, \tag{9}$$

which implies

$$x_{n+1} = x_n - \left(1 \pm \frac{1}{m}\right)\frac{f(x_n)}{f'(x_n)};\ m \geq 2, m \in \mathrm{N}, \tag{10}$$

where $x_n > x^*$ and n ($n = 0,1,2,3,...$) indicate number of iterations.

(B) Using Taylor's series expansion and neglecting terms containing O (h^3) and higher as

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) + O(h^3), \tag{11}$$

equation (4) becomes

$$m^2[f'^2(x_0) + f(x_0)f''(x_0)]h^2 + [-2m^2 f(x_0)f'(x_0)]h + (m^2 - 1)f^2(x_0) = 0. \tag{12}$$

Considering

$$A = m^2[f'^2(x_0) + f(x_0)f''(x_0)],$$
$$B = -2m^2 f(x_0)f'(x_0),$$
$$C = (m^2 - 1)f^2(x_0),$$

equation (12) becomes

$$Ah^2 + Bh + C = 0\,, \tag{13}$$

which on solving for h gives

$$h = \frac{f(x_0)f'(x_0) \pm \sqrt{\frac{1}{m^2}f^2(x_0)f'^2(x_0) - (1 - \frac{1}{m^2})f^3(x_0)f''(x_0)}}{f(x_0)f''(x_0) + f'^2(x_0)}. \tag{14}$$

Therefore, from equation (3) the first approximation can be obtained as

$$x_1 = x_0 - \frac{f(x_0)f'(x_0) \pm \sqrt{\frac{1}{m^2} f^2(x_0) f'^2(x_0) - (1-\frac{1}{m^2}) f^3(x_0) f''(x_0)}}{f(x_0)f''(x_0) + f'^2(x_0)}. \quad (15)$$

The general formula for successive approximation is , therefore, given by

$$x_{n+1} = x_n - h, \quad (16)$$

which implies

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n) \pm \sqrt{\frac{1}{m^2} f^2(x_n) f'^2(x_n) - (1-\frac{1}{m^2}) f^3(x_n) f''(x_n)}}{f(x_n)f''(x_n) + f'^2(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (17)$$

where $x_n > x^*$.

Case II When $x_0 < x^*$

In this case

$$x_1 = x_0 + h, \quad (18)$$

be the first approximation to the required root of equation (1).

Then equation (2) becomes

$$\{f(x_0 + h)\}^2 = \frac{[f(x_0)]^2}{m^2}. \quad (19)$$

(A) By following the same procedure as in Case I (A) , the general formula for successive approximation can be obtained as

$$x_{n+1} = x_n + \left(-1 \pm \frac{1}{m}\right)\frac{f(x_n)}{f'(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (20)$$

where $x_n < x^*$ and $x_{n+1} = x_n + h$.

(B) By following the same procedure as in Case I (B), the general formula for successive approximation can be obtained as

$$x_{n+1} = x_n + \frac{-f(x_n)f'(x_n) \pm \sqrt{\frac{1}{m^2} f^2(x_n) f'^2(x_n) - (1-\frac{1}{m^2}) f^3(x_n) f''(x_n)}}{f(x_n)f''(x_n) + f'^2(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (21)$$

where $x_n < x^*$ and $x_{n+1} = x_n + h$.

Combining formulae (10) and (20) yields the iterative scheme

$$x_{n+1} = x_n - \left(1 \pm \frac{1}{m}\right)\frac{f(x_n)}{f'(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (22)$$

which is true for any initial guess x_0 ; that is, whether $x_0 > x^*$ or $x_0 < x^*$.

Scheme (22) implies two different formulas as

$$x_{n+1} = x_n - \left(1 - \frac{1}{m}\right)\frac{f(x_n)}{f'(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (23)$$

$$x_{n+1} = x_n - \left(1 + \frac{1}{m}\right)\frac{f(x_n)}{f'(x_n)}; m \geq 2, m \in \mathrm{N}. \quad (24)$$

Similarly, combining formulae (17) and (21) yields

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n) \pm \sqrt{\frac{1}{m^2} f^2(x_n) f'^2(x_n) - (1-\frac{1}{m^2}) f^3(x_n) f''(x_n)}}{f(x_n)f''(x_n) + f'^2(x_n)}; m \geq 2, m \in \mathrm{N}, \quad (25)$$

which is true for any initial guess x_0.

3. Convergence Analysis

Let ε_n be the error in the n^{th} iteration then

$$\varepsilon_n = x^* - x_n. \tag{26}$$

Similarly, let ε_{n+1} be the error in the $(n+1)^{th}$ iteration then

$$\varepsilon_{n+1} = x^* - x_{n+1}. \tag{27}$$

Convergence analysis of scheme (22)

Substituting equations (26), (27) in (22) yields

$$x^* - \varepsilon_{n+1} = x^* - \varepsilon_n - \left(1 \pm \frac{1}{m}\right) \frac{f(x^* - \varepsilon_n)}{f'(x^* - \varepsilon_n)}, \tag{28}$$

which implies

$$\varepsilon_{n+1} = \varepsilon_n + \left(1 \pm \frac{1}{m}\right) \frac{\left[-\varepsilon_n f'(x^*) + \frac{\varepsilon_n^2}{2!} f''(x^*) - \ldots\right]}{\left[f'(x^*) - \varepsilon_n f''(x^*) + \frac{\varepsilon_n^2}{2!} f'''(x^*) - \ldots\right]}, \tag{29}$$

since $f(x^*)=0$.

On simplification of equation (29) , one obtains

$$\begin{aligned} \varepsilon_{n+1} &\approx \pm \frac{1}{m} \varepsilon_n + \left(-\frac{1}{2} \pm \frac{1}{2m}\right) \varepsilon_n^2 \frac{f'(x^*)}{f''(x^*)} + \ldots \\ &\approx \varepsilon_n (\text{Finite quantity}). \end{aligned} \tag{30}$$

Thus, the scheme (22) converges linearly. However, it can be observed that when m is large then the order of convergence is 2 (Quadratically) and the scheme resemblance to Newton's formula.

Convergence analysis of scheme (25)

Substituting equations (26), (27) in (25), the separate simplifications of each term in (25) gives

$$f(x_n) f'(x_n) = f(x^* - \varepsilon_n) f'(x^* - \varepsilon_n) = -\varepsilon_n f'^2 + \frac{3}{2} \varepsilon_n^2 f' f'' - \frac{\varepsilon_n^3}{2} f''^2 - \ldots. \tag{31}$$

$$[f(x_n) f'(x_n)]^2 = \varepsilon_n^2 f'^4 + \frac{13}{4} \varepsilon_n^4 f'^2 f''^2 + \frac{1}{4} \varepsilon_n^6 f''^4 - 3\varepsilon_n^3 f'^3 f'' - \frac{3}{2} \varepsilon_n^5 f' f''^3 - \ldots. \tag{32}$$

$$f^3(x_n) f''(x_n) = -\varepsilon_n^3 f'^3 f'' + \frac{1}{8} \varepsilon_n^6 f''^4 + \frac{3}{2} \varepsilon_n^4 f'^2 f''^2 - \frac{3}{4} \varepsilon_n^5 f' f''^3 - \ldots. \tag{33}$$

$$f(x_n) f''(x_n) = -\varepsilon_n f' f'' + \frac{1}{2} \varepsilon_n^2 f''^2 - \ldots. \tag{34}$$

$$f'^2(x_n) = f'^2 - 2\varepsilon_n f' f'' + \varepsilon_n^2 f''^2 - \ldots. \tag{35}$$

The term under square root sign in (25) is given by

$$\begin{aligned} \frac{1}{m^2} f^2 f'^2 - \left(1 - \frac{1}{m^2}\right) f^3 f'' &= \frac{1}{m^2} \left(\varepsilon_n^2 f'^4 + \frac{19}{4} \varepsilon_n^4 f'^2 f''^2 - 4\varepsilon_n^3 f'^3 f'' - \ldots \right) \\ &\quad - \left(-\varepsilon_n^3 f'^3 f'' + \frac{3}{2} \varepsilon_n^4 f'^2 f''^2 - \ldots \right), \end{aligned} \tag{36}$$

which is obtained by neglecting $O(\varepsilon_n^5)$, and higher.

The denominator term in (25) is given by

$$f f'' + f'^2 = f'^2 - 3\varepsilon_n f' f'' + \frac{3}{2} \varepsilon_n^2 f''^2 - \ldots \tag{37}$$

The second term in (25) is given by

$$\frac{f\,f'}{f\,f''+f'^2}=\frac{-\varepsilon_n f'^2+\frac{3}{2}\varepsilon_n^2 f'\,f''-\frac{1}{2}\varepsilon_n^3 f''^2-\ldots}{f'^2-3\varepsilon_n f'\,f''+\frac{3}{2}\varepsilon_n^2 f''^2-\ldots} \tag{38}$$

On right hand side divide by f'^2 to both numerator and denominator, one obtains

$$\frac{f\,f'}{f\,f''+f'^2}=\frac{-\varepsilon_n+\frac{3}{2}\varepsilon_n^2\frac{f''}{f'}-\frac{1}{2}\varepsilon_n^3\frac{f''^2}{f'^2}-\ldots}{1-3\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}-\ldots} \tag{39}$$

The third term in (25) is given by

$$\frac{\sqrt{\frac{1}{m^2}f^2f'^2-\left(1-\frac{1}{m^2}\right)f^3f''}}{f\,f''+f'^2}=\frac{\varepsilon_n\sqrt{\frac{1}{m^2}\left(1+\frac{19}{4}\varepsilon_n^2\frac{f''^2}{f'^2}-4\varepsilon_n\frac{f''}{f'}-\ldots\right)-\left(-\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}+\ldots\right)}}{1-3\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}-\ldots} \tag{40}$$

Using equations (26),(27) in (25) , and using equations (31)-(40), one obtains on simplification

$$\varepsilon_{n+1}=\frac{-\frac{3}{2}\varepsilon_n^2\frac{f''}{f'}+\varepsilon_n^3\frac{f''^2}{f'^2}-\ldots\pm\varepsilon_n\sqrt{\frac{1}{m^2}\left(1+\frac{19}{4}\varepsilon_n^2\frac{f''^2}{f'^2}-4\varepsilon_n\frac{f''}{f'}-\ldots\right)-\left(-\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}+\ldots\right)}}{1-3\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}-\ldots} \tag{41}$$

$$\Rightarrow \varepsilon_{n+1}\approx-\frac{3}{2}\varepsilon_n^2\frac{f''}{f'}+\varepsilon_n^3\frac{f''^2}{f'^2}-\ldots\pm\varepsilon_n\sqrt{\frac{1}{m^2}\left(1+\frac{19}{4}\varepsilon_n^2\frac{f''^2}{f'^2}-4\varepsilon_n\frac{f''}{f'}-\ldots\right)-\left(-\varepsilon_n\frac{f''}{f'}+\frac{3}{2}\varepsilon_n^2\frac{f''^2}{f'^2}+\ldots\right)} \tag{42}$$

$$\Rightarrow \varepsilon_{n+1}\approx\varepsilon_n(\text{Finite quantity}). \tag{43}$$

Thus, the scheme (25) converges linearly. However, it can be observed from equation (42) that when m is large then the order of convergence is 3/2.

4. Discussion of some Examples

As the proposed iterative scheme (22) converges quadratically when m is large, so a comparison of this scheme with Newton's method is given.

It is observed from Table 1 that by using Newton's method, the answer correct up to four decimal places is 1.3247 which is obtained in 5 iterations whereas using our method the same answer is obtained when

(1) m = 38 for formula (23) (Column 3) ; that is , the number of iterations remains 5 as in Newton's case.

(2) m = 10 for formula (24) (column 4); that is , the number of iterations remains 5 as in Newton's case.

It is observed from Table 2 that by using Newton's method, the answer correct up to four decimal places is 1.0000 which is obtained in 14 iterations whereas using our method the same answer is obtained when

(1) m = 46 for formula (23) (column 3) ; that is , the number of iterations remains 14 as in Newton's case.

(2) m = 2 for formula (24) (column 4) ; that is , the number of iterations required are 8 which are less than Newton's case where it requires 14. So, faster convergence is obtained.

By observing both the examples carefully, it is concluded that the new proposed iterative scheme is comparable with Newton's scheme. Moreover, for example in Table 2, faster convergence is obtained for the formula (24) in column 4.

Table 1. Comparative study of our scheme (22) with Newton's formula for $x^3-x-1=0$

(1)	(2)	(3)	(4)	(5)	(6)
Initial guess value x_0	Values of m	Number of iterations required by using formula (23) to get answer as in column (6)	Number of iterations required by using formula (24) to get answer as in column (6)	Number of iterations by using Newton's formula	Solution of the given equation correct up to 4 decimal places
2	2	16	11	5	1.3247
	3	11	10		
	4	9	8		
	10	7	5		
	11	7	5		
	12	6	5		
	20	6	5		
	30	6	5		
	38	5	5		

Table 2. Comparative study of our scheme (22) with Newton's formula for $x^3-x^2-x+1=0$

(1)	(2)	(3)	(4)	(5)	(6)
Initial guess value x_0	Values of m	Number of iterations required by using formula (23) to get answer as in column (6)	Number of iterations required by using formula (24) to get answer as in column (6)	Number of iterations by using Newton's formula	Solution of the given equation correct up to 4 decimal places
1.5	2	32	8	14	1.0000
	3	23	10		
	4	20	10		
	10	16	12		
	20	15	13		
	30	15	14		
	40	15	14		
	46	14	14		

5. Conclusions

The present paper proposes new iterative schemes for finding a root of a nonlinear algebraic equation $f(x) = 0$. The discussion Section 4 shows the comparison of our method with Newton's Method. The following are the major conclusions obtained:

(1) The scheme (22) converges linearly. However, when m is large, the order of convergence is 2 and the scheme resemblance to Newton's formula.

(2) The necessary condition for convergence of the scheme (25) is

$$f^2(x_n)f'^2(x_n)-(m^2-1)f^3(x_n)f''(x_n)\geq 0 .$$

Also, the condition of validity of the scheme is

$$f(x_n)f''(x_n)+f'^2(x_n)\neq 0 ,$$

which implies

$$f''(x_n)\neq 0 \text{ and } f'(x_n)\neq 0 .$$

(3) The scheme in (25) works even if

$$f'(x_n)=0 \text{ provided } f''(x_n)\neq 0.$$

(4) The scheme (25) has linear convergence. However, when m is large the order of convergence is 3/2.

(5) There is no restriction on initial guess x_0 .

(6) The beauty of the present schemes (22) and (25) are of choosing value of m, which is in our control.

(7) The example discussed in Table 2 shows the faster convergence by our scheme than Newton's method.

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