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Derivation of the Modified Bi-quintic B-spline Base Functions: An Application to Poisson Equation

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Abstract In this paper, the bi-quintic B-spline base functions have been modified on a general two dimensional problem. A special form of the two dimensional problem has been considered as its application and has been solved by the Galerkin finite element method using the modified bi-quintic B-spline base functions. The computed results obtained by the present method have also been compared with the results available in the literature.

Keywords Galerkin Finite Element Method, Bi-quintic B-splines, Two-dimensional B-splines, Modified bi-quintic B-splines, Two-dimensional Poisson Equation

MSC classification: 65N30, 65N30, 65D07, 74S05

1. Introduction

The finite element method is a widely used method for solving both ordinary and partial differential equations. For a long time, the finite element method has been applied by many researchers to physics, solid and fluid mechanics, engineering, medicine, and so on[1]-[9]. The main idea behind the finite element method is to divide the whole region of the given problem into an equivalent system of finite elements with associated nodes and to choose the most appropriate element type to model most closely the actual physical behavior. Thus, by means of the finite element method, a huge problem is converted into many solvable small problems. Those elements must be made small enough to give usable results and yet large enough to reduce computational effort [3].

B-spline finite element method has been previously and widely used to obtain very accurate solutions for partial differential equations in one dimensional problem [2]-[8], [10]-[15]. Nevertheless, developments in computational techniques and computer technology have shown that the method can also be effectively used to obtain accurate approximate solutions for two-dimensional problems. In this paper, the bi-quintic B-splines on the boundary of the model problem have been modified and then they have been used to solve a typical two--dimensional problem. For two-dimensional problems, there are some works in the literature.[2, 7, 9, 16].

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In this study, we have used bi-quintic B-splines which are only modified on the boundary of the problem as basis functions and rectangles as element shapes for a two-dimensional problem. The modified bi-quintic B-spline functions have been constructed for two dimensional problems and applied to a typical two-dimensional problem of the form $-\Delta u = f(x, y)$. First of all, we try a bi-quintic B-spline function of the form

$$u_{nm}(x, y) = \sum_{i=-2}^{n+2} \sum_{j=-2}^{m+2} \alpha_{ij} B_i(x) B_j(y)$$

as an approximation solution to this type of equations. In order to find an approximate solution in the above form to the considered problem by the Galerkin method, first of all we have to redefine the B-spline basis functions into a new set of functions, namely modified bi-quintic B-spline functions. This redefining process was successfully applied to cubic B-spline functions to obtain modified bi-cubic B-spline functions [2]. The newly obtained set of modified functions is identically zero on the boundary of the given problem. It should be pointed out that this process is necessary, since B-spline basis functions $B_i(x)(i = -2(1)2,\ (n-2)(1)(n+2))$ in the x -direction and the B-spline basis functions $B_j(y)(j = -2(1)2,\ (m-2)(1)(m+2))$ in the y -direction are not zero on the boundary of the given problem. After the redefining process of the basis functions, we can now try the modified bi-quintic B-spline functions of the form

$$u_{nm}(x, y) = \varphi(x, y) + \sum_{i=-1}^{n+1} \sum_{j=1}^{m+1} \alpha_{ij}\, \tilde{B}_i(x) \tilde{B}_j(y)$$

as its approximate solution, where $\tilde{B}_i(x)$ and $\tilde{B}_j(y)$ are quintic B-splines modified only on the boundary in the x and y - directions, respectively. Since all of the modified quintic B-splines are zero on the boundary of the given problem, non-homogenous boundary conditions are satisfied by the term $\varphi(x,y)$.

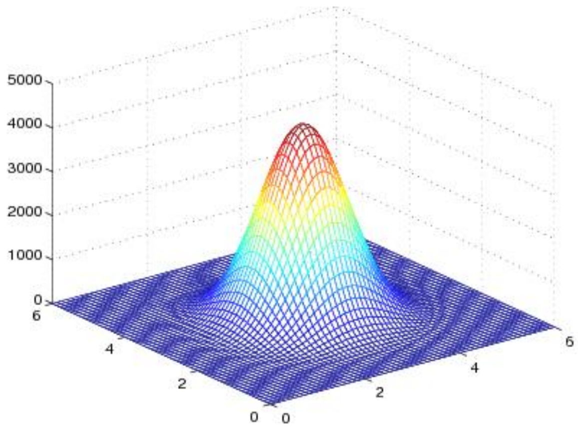

Figure 1. The bi-quintic B-spline B_{33} centred on (3,3) covering 36 finite elements of side 1

2. Derivation of the Modified Bi-Quintic B-splines

2.1. Bi-quintic B-spline Element

Suppose that a rectangular grid is subdivided into a number of uniform rectangular finite elements of sides h_x and h_y by the knots (x_i, y_j) where $0 \le i \le n,$ $0 \le j \le m$. An approximation $u_{nm}(x,y)$ with quintic B-spline functions to $u(x,y)$ is of the form

$$B_i(x)=\frac{1}{h^5}\begin{cases} (x-x_{i-3})^5, & [x_{i-3},x_{i-2}],\\ (x-x_{i-3})^5-6(x-x_{i-2})^5, & [x_{i-2},x_{i-1}],\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5, & [x_{i-1},x_i],\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5-20(x-x_i)^5, & [x_i,x_{i+1}],\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5-20(x-x_i)^5+15(x-x_{i+1})^5, & [x_{i+1},x_{i+2}],\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5-20(x-x_i)^5+15(x-x_{i+1})^5-6(x-x_{i+2})^5, & [x_{i+2},x_{i+3}],\\ 0, & otherwise. \end{cases}$$

$B_j(y)$ can easily be found by replacing i with j and x with y. Fig.1 depicts a region where $h_x = h_y = 1$, so that it is divided into finite elements by the integer knots (i,j) , and a single bi-quintic B-spline B_{33} which peaks on the point $(3,3)$ and also covers a total of 36 square elements. When the entire set of bi-quintic splines B_{ij} , each of which peaks on a knot (i,j) , where $0 \le i \le 6,\ \ 0 \le j \le 6$, are added to this figure, a total of 36 splines cover each finite element [8].

$$u_{nm}(x,y)=\sum_{i=-2}^{n+2}\sum_{j=-2}^{m+2}\alpha_{ij}B_{ij}(x,y) \qquad (1)$$

where α_{ij} 's are the amplitudes of bi-quintic B-splines $B_{ij}(x,y)$ given by

$$B_{ij}(x,y)=B_i(x)B_j(y) \qquad (2)$$

and $B_i(x)$ is defined as[11]

2.2. A Modified Two-dimensional Quintic B-spline Element

To show how to modify bi-quintic splines on the boundary, we consider the two-dimensional general linear problem given in the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial y^2}+c(x,y)\frac{\partial u}{\partial x} +d(x,y)\frac{\partial u}{\partial y}+e(x,y)u=-f(x,y) \qquad (3)$$

subject to boundary conditions

$$u(x,y_0)=g_1(x), \quad x_0 \le x \le x_n \qquad (4)$$

$$u(x,y_m)=g_2(x), \quad x_0 \le x \le x_n \qquad (5)$$

$$u(x_0,y)=h_1(y), \quad y_0 \le y \le x_m \qquad (6)$$

$$u(x_n,y)=h_2(y), \quad y_0 \le y \le x_m \qquad (7)$$

where $x\in[x_0,x_n], y\in[y_0,y_m]$ and $g_1(x),$ $g_2(x),\ h_1(y),\ \ h_2(y)$ are given functions, D is a rectangular region in R^2 with boundary ∂D .

Now, it is supposed that both the x -space variable domain and y -space variable domain of the system (3)-(7) are divided into n and m sub-intervals, respectively, by the set of $n+1$ distinct grid points x_i $(i=0(1)n)$ and $m+1$ distinct grid points y_j $(j=0(1)m)$ such that

$0=x_0<x_1<...<x_n=1$ and $0=y_0<y_1<...<y_m=1.$

Since we use quintic B-splines, a quintic B-spline covers six consecutive elements, we add ten additional grid points x_{-5} , x_{-4} , x_{-3} , x_{-2} , x_{-1} , x_{n+1} , x_{n+2} , x_{n+3} , x_{n+5} in the x -direction and ten additional grid points y_{-5} ,

y_{-4} , y_{-3} , y_{-2} , y_{-1} , y_{m+1} , y_{m+2} , y_{m+3} , y_{m+4} , y_{m+5} in the y -direction such that

$$h_x = x_{n+1} - x_n, \quad n = -5,-4,-3,-2,-1,0$$

$$h_y = y_{m+1} - y_m, \quad m = -5,-4,-3,-2,-1,0$$

$$h_x = x_{n+1} - x_n, \quad n = n+4, n+3, n+2, n+1, n, n-1$$

$$h_y = y_{m+1} - y_m, \quad m = m+4, m+3, m+2, m+1, m, m-1.$$

To obtain an approximate solution in the form of Eq. (1) to the model problem given by Eqs.(3)-(7) with the Galerkin method, we do need to redefine the basis functions into a new set of basis functions which all vanish on ∂D . The redefining process of the basis functions is done in the following three steps.

Step1. The approximate solution $u_{nm}(x,y)$ given by Eq. (1) can also be written as[2]

$$u_{nm}(x,y) = \sum_{i=-2}^{n+2} \gamma_i(y) B_i(x) \tag{8}$$

where

$$\gamma_i(y) = \sum_{j=-2}^{m+2} \alpha_{ij} B_j(y). \tag{9}$$

Allowing the approximate solution $u_{nm}(x,y)$ given by Eq. (8) to satisfy the boundary conditions (6) and (7) and eliminating $\gamma_{-2}(y)$ and $\gamma_{n+2}(y)$ from the resulting equations, we obtain

$$u_{nm}(x,y) = \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \sum_{i=-1}^{n+1} \gamma_i(y) \tilde{B}_i(x) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) \tag{10}$$

where

$$\tilde{B}_i(x) = B_i(x) - \frac{B_i(x_0)}{B_{-2}(x_0)} B_{-2}(x) \quad i = -1,0,1,2 \tag{11}$$

$$\tilde{B}_i(x) = B_i(x), \quad i = 3(1)n-3 \tag{12}$$

$$\tilde{B}_i(x) = B_i(x) - \frac{B_i(x_n)}{B_{n+2}(x_n)} B_{n+2}(x) \tag{13}$$

$$i = n-2, n-1, n, n+1$$

Step 2. By evaluating the expression $\gamma_i(y)$ given by Eq. (9) at y_0 and y_m and eliminating $\alpha_{i,-2}$ and $\alpha_{i,m+2}$ from the resulting equations, we obtain the following expression for $\gamma_i(y)$:

$$\gamma_i(y) = \frac{B_{-2}(y)}{B_{-2}(y_0)} \gamma_i(y_0) + \sum_{j=-1}^{m+1} \alpha_{ij}(y) \tilde{B}_j(y) + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \gamma_i(y_m) \tag{14}$$

where

$$\tilde{B}_j(y) = B_j(y) - \frac{B_j(y_0)}{B_{-2}(y_0)} B_{-2}(y) \quad i = -1,0,1,2 \tag{15}$$

$$\tilde{B}_j(y) = B_j(y), \quad j = 3(1)m-3 \tag{16}$$

$$\tilde{B}_j(y) = B_j(y) - \frac{B_j(y_m)}{B_{m+2}(y_m)} B_{m+2}(y) \tag{17}$$

$$i = m-2, m-1, m, m+1$$

Step 3. Finally, substituting $\gamma_i(y)$ in Eq. (14) into Eq. (10) and allowing the resulting equation to satisfy the boundary conditions (4) and (5), we obtain

$$u_{nm}(x,y) = \varphi_1(x,y) + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij} \tilde{B}_i(x) \tilde{B}_j(y) \tag{18}$$

where

$$\begin{aligned} \phi_1(x,y) = {} & \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) + \frac{B_{-2}(y)}{B_{-2}(y_0)} g_1(x) \\ & + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} g_2(x) \\ & - \frac{B_{-2}(y)}{B_{-2}(y_0)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y_0) \right] \\ & - \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y_m) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y_m) \right] \end{aligned} \tag{19}$$

Applying exactly the same above steps to the approximate solution (1) to the problem given by Eqs. (3)-(7) again, but now writing the approximate solution as[2]

$$u_{nm}(x,y) = \sum_{j=-2}^{m+2} \delta_i(x) B_j(y) \tag{20}$$

where

$$\delta_j(x) = \sum_{j=-2}^{n+2} \alpha_{ij} B_i(x). \tag{21}$$

we obtain

$$u_{nm}(x,y) = \varphi_2(x,y) + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij} \tilde{B}_i(x) \tilde{B}_j(y) \tag{22}$$

Where

$$\begin{aligned} \phi_2(x,y) = {} & \frac{B_{-2}(x)}{B_{-2}(x_0)} h_1(y) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} h_2(y) \\ & + \frac{B_{-2}(y)}{B_{-2}(y_0)} g_1(x) + \frac{B_{m+2}(y)}{B_{m+2}(y_m)} g_2(x) \\ & - \frac{B_{-2}(y)}{B_{-2}(y_0)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} g_1(x_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} g_1(x_n) \right] \\ & - \frac{B_{m+2}(y)}{B_{m+2}(y_m)} \left[\frac{B_{-2}(x)}{B_{-2}(x_0)} g_2(x_0) + \frac{B_{n+2}(x)}{B_{n+2}(x_n)} g_2(x_n) \right]. \end{aligned} \tag{23}$$

By taking the average of Eq. (19) and Eq. (23), we finally obtain the general approximation $u_{nm}(x,y)$ to $u(x,y)$ of the form

$$\begin{aligned} u_{nm}(x,y) &= \frac{\varphi_1(x,y) + \varphi_2(x,y)}{2} + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij} \tilde{B}_i(x) \tilde{B}_j(y) \\ u_{nm}(x,y) &= \varphi(x,y) + \sum_{i=-1}^{n+1} \sum_{j=-1}^{m+1} \alpha_{ij} \tilde{B}_i(x) \tilde{B}_j(y) \end{aligned} \tag{24}$$

where the new set of basis functions are $\tilde{B}_i(x) \tilde{B}_j(y)$ for $i = -1(1)n+1$, $j = -1(1)m+1$, which all vanish on ∂D

and $\varphi(x,y)$ given in Eq. (24) satisfies the boundary conditions given by Eqs. (4) - (7). For simplicity, the modified quintic B-splines are shown in Fig.2 for only 4 elements.

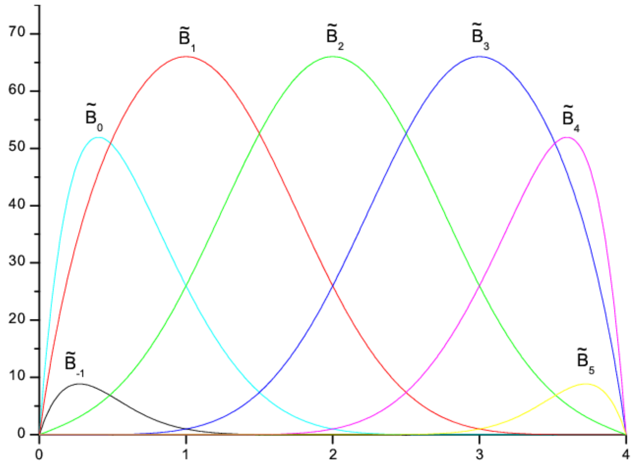

Figure 2. Modified B-spline functions $\tilde{B}_i(x), i=-1(1)5.$

3. Numerical Example

There are some researches about two-dimensional Poisson equation in the literature. Wu and Zhang [7] have obtained numerical solutions of Dirichlet problem for two-dimensional Poisson equation using the Galerkin quartic B-spline finite element method. Schaerer et al.[17] have presented Multilevel Schwarz Shooting method for the numerical solution of the Poisson equation using a five point finite difference molecule, and subject to Dirichlet boundary conditions, which arises in two dimensional incompressible flow simulations. Ahmed and Monaquel[18] have developed multi-grid method with compact finite difference method of 9-point sixth order for two-dimensional Poisson equation and showed the efficiency of the method with numerical solutions.

In this section, we have applied the Galerkin method using the modified bi-quintic B-spline basis functions to obtain the numerical solutions of the two dimensional Poisson problem given as

$$\Delta u = 5(x-1)[2\cos(5y)-5(y-1)\sin(5y)]$$
$$= f(x,y), \quad (x,y)\in D$$
$$u(x,y_0)=g_1(x)=0, \quad x_0\le x\le x_n,$$
$$u(x,y_m)=g_2(x)=0, \quad x_0\le x\le x_n,$$
$$u(x_0,y)=h_1(y)=0, \quad y_0\le y\le x_m,$$
$$u(x_n,y)=h_2(y)=(1-y)\sin(5y), \quad y_0\le y\le x_m,$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ and $D=[0,1]\times[0,1]$.

The exact solution of this problem is $u(x,y)=(1-x)(1-y)\sin(5y)$ [7].

To apply the Galerkin method with bi-quintic B-splines, we first need to find the weak formulation of the problem. Its weak form is given by the integral equation

$$\iint\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\Psi(x,y)dxdy=0 \qquad (25)$$

where $\Psi(x,y)$ is the weight function taken as modified bi-quintic B-spline functions. Using the Green Theorem and the values of the modified bi-quintic B-spline functions on the boundary conditions, we obtain the weak form of the problem as

$$\iint\left(\frac{\partial u}{\partial x}\frac{\partial \Psi}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial \Psi}{\partial y}\right)dxdy=\iint f(x,y)\Psi(x,y)dxdy. \quad (26)$$

For the sake of simplicity, we divide the region D into $4\times4=16$ elements and show the procedure in detail. Thus, we have 16 squares with sides of $1/4$, and $h_x=h_y=h=1/4$. We use the following numeration of the region D with 16 elements in the following processes.

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

Since each part of a quintic B-spline should be multiplied by the same coefficient to satisfy the continuity condition, the modified quintic B-splines $\tilde{B}_i(x)$ and $\tilde{B}_j(y)$ are multiplied by local coefficients α_i and β_j $(i,j=-1(1)5)$, respectively. Therefore, we have total $7\times7=49$ global coefficients to be computed. The global coefficients and their corresponding local coefficients are determined as

$$A_k=\alpha_i\ \beta_j \quad \text{for} \quad i,j=-1(1)5 \text{ and } k=1(1)49.$$

Thus, we obtain the approximations for each element by writing the linear combination of $\tilde{B}_i(x)$ and $\tilde{B}_j(y)$ lying on the related element in Eq. (26).

By applying the Galerkin method with $\tilde{B}_i(x)\tilde{B}_j(y)$ $i=-1(1)n+1,$ and $j=-1(1)m+1,$ as basis functions and with an approximation $u_{nm}(x,y)$ given by Eq.(26) to the model problem, in a similar way given in Ref.[2], we get the following algebraic system:

$$A\delta=f,$$

where

$$A=\left[A_{i_l j_l}\right],$$

$$A_{i_l j_l}=\int_{x_0}^{x_n}\int_{y_0}^{y_m}\left[\frac{d\tilde{B}_i(x)}{dx}\frac{d\tilde{B}_k(x)}{dx}\tilde{B}_j(y)\tilde{B}_l(y)\right.$$
$$\left.+\frac{d\tilde{B}_j(y)}{dy}\frac{d\tilde{B}_l(y)}{dy}\tilde{B}_i(x)d\tilde{B}_k(x)\right]dxdy,$$

$$i_l = j(n+1)+i,\quad j_l = l(n+1)+k,$$
$$\text{for } i,k = -1(1)n+1,\ j,l = -1(1)m+1,$$
$$f = \left[f_{i_l} \right],$$
$$f_{i_l} = \int_{x_0}^{x_n}\int_{y_0}^{y_m} \left[f(x,y)\tilde{B}_i(x)\tilde{B}_j(y) - \frac{\partial\varphi(x,y)}{\partial x}\frac{d\tilde{B}_i(x)}{dx}\tilde{B}_j(y) - \frac{\partial\varphi(x,y)}{\partial y}\frac{d\tilde{B}_j(y)}{dy}\tilde{B}_i(x) \right] dxdy,$$
$$i_l = j(n+1)+i,\quad \text{for}\quad i=-1(1)n+1,\ j=-1(1)m+1$$

and δ is the coefficient vector to be computed.

Assembling all contributions from all elements, we reach the global combined matrix. Since the modified bi-quintic B-splines have been altered so as to meet the boundary conditions beforehand, the resulting equations are solved without imposing the boundary conditions at the end of the element assembly process. By solving the system of equations obtained in the global combined matrix, we evaluate the values of global variables $A_1 - A_{49}$.

Table 1 displays the obtained numerical and exact solutions of the problem with those given in Ref.[7]. It is clearly seen that the obtained numerical solutions are in very good agreement with exact ones. They are also much better than those given in Ref.[7] using quartic B-spline finite element and finite difference methods. To show how good and accurate the numerical solutions of the problem, a comparison of the obtained numerical and exact solutions at some different points are displayed in Table 2.

Table 1. Comparison of the numerical solutions obtained by the present method with results from Ref. [7] and the exact solution at some points

Present			[7]		
Quintic			Quartic	Finite Difference	Exact
(x,y)	(4×4)	(10×10)			
(0.25,0.25)	0.533970	0.533803	0.523514	0.543869	0.533804
(0.25,0.50)	0.224352	0.224426	0.229845	0.220623	0.224400
(0.25,0.75)	-0.107419	-0.107167	-0.098704	-0.117514	-0.107168
(0.50,0.25)	0.355980	0.355869	0.345257	0.364192	0.355869
(0.50,0.50)	0.149568	0.149618	0.152811	0.146201	0.149618
(0.50,0.75)	-0.071613	-0.071445	-0.064679	-0.080613	-0.071445
(0.75,0.25)	0.177990	0.177935	0.173169	0.182276	0.177935
(0.75,0.50)	0.074784	0.074809	0.076689	0.072922	0.074809
(0.75,0.75)	-0.035806	-0.035722	-0.032649	-0.040688	-0.035723

Table 2. Comparison of the numerical solutions obtained by the present method with a partition of 4×4=16 and 10×10=100 elements with exact ones at some points

	Present		Exact
(x,y)	(4×4)	(10×10)	
(0.1,0.1)	0.388262	0.388335	0.388335
(0.2,0.2)	0.538588	0.538542	0.538541
(0.3,0.3)	0.488891	0.488773	0.488773
(0.4,0.4)	0.327186	0.327347	0.327347
(0.5,0.5)	0.149568	0.149618	0.149618
(0.6,0.6)	0.022734	0.022579	0.022579
(0.7,0.7)	-0.031608	-0.031571	-0.031571
(0.8,0.8)	-0.030308	-0.030272	-0.030272
(0.9,0.9)	-0.009757	-0.009775	-0.009775
(0.0,1.0)	0.000000	0.000000	0.000000
(0.1,0.9)	-0.087821	-0.087978	-0.087978
(0.2,0.8)	-0.121231	-0.121089	-0.121088
(0.3,0.7)	-0.073754	-0.073665	-0.073665
(0.4,0.6)	0.034100	0.033868	0.033869
(0.6,0.4)	0.218123	0.218231	0.218231
(0.7,0.3)	0.209524	0.209474	0.209474
(0.8,0.2)	0.134648	0.134636	0.134635
(0.9,0.1)	0.043140	0.043148	0.043148
(1.0,0.0)	0.000000	0.000000	0.000000

Table 3. The error norms L_2 and L_∞ of the model problem for various numbers of elements and B-spline base functions

	# of elements	L_2	L_∞
Cubic [19]	2×2	6.25550×10^{-2}	3.26868×10^{-2}
	4×4	6.05939×10^{-3}	4.56680×10^{-3}
	10×10	2.79940×10^{-4}	5.50414×10^{-4}
Quintic	4×4	4.29559×10^{-4}	3.14455×10^{-4}
	10×10	1.04642×10^{-6}	9.40069×10^{-7}

In order to measure how good the numerical solutions obtained by the Galerkin Finite Element Method with the modified bi-quintic B-spline basis functions, the error norms L_2 and L_∞ defined as

$$L_2 = |u - u_{nm}|_2 = \frac{\sqrt{\sum_{i=1}^{n_{i_p}} |u_i - (u_{nm})_i|^2}}{\sqrt{\sum_{i=1}^{n_{i_p}} |u_i|^2}},$$

$$L_\infty = |u - u_{nm}|_\infty = \max_{0<i<n_{i_p}} |u_i - (u_{nm})_i|$$

are computed and given in Table 3. Here n_{i_p} is the number of inner points, u_i and $(u_{nm})_i$ are the exact and approximate solutions at the point i, respectively. The error norms L_2 and L_∞ are computed by taking the values u_i and $(u_{nm})_i$ at n_{i_p} points obtained by dividing the region $D=[0,1]\times[0,1]$ into equal elements in the directions x and y. As seen from Table 3, the approximate solution becomes better as the number of elements increase.

Moreover, in order to demonstrate how good the numerical solutions obtained by the present method exhibit the correct physical characteristic of the problem, we also give the graphs in Figs. 3, 4 and 5 which illustrate the exact and numerical solutions with 16 and 100 partitions of the model problem, respectively. It is clearly seen from the figures that both graphs are nearly indistinguishable since both solutions are in very good agreement with each other.

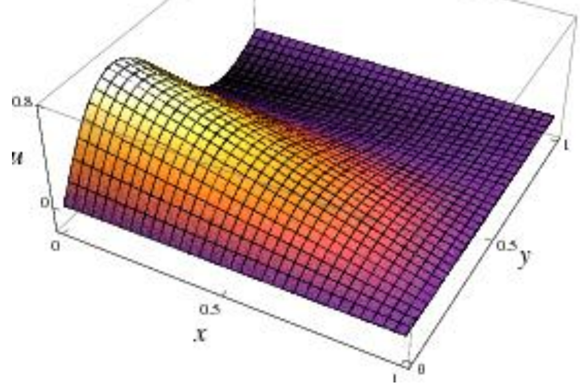

Figure 3. Exact solution

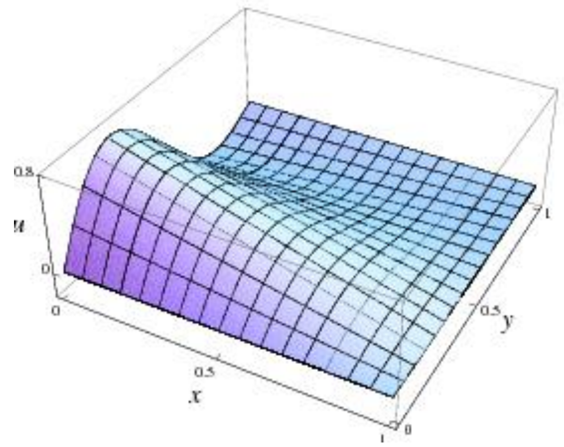

Figure 4. Numerical solution with 16 elements partitions

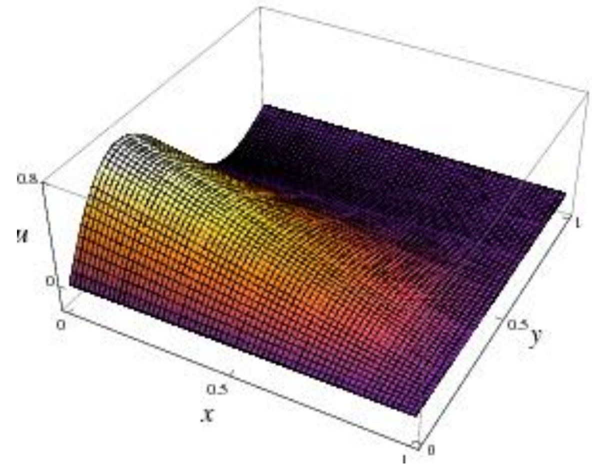

Figure 5. Numerical solution with 100 elements partitions

4. Conclusions

In this paper, the modified bi-quintic B-spline finite element method is proposed and successfully applied to two dimensional Poisson problem to obtain its numerical solutions. The agreement between our numerical results and the exact ones is satisfactorily good. The obtained numerical results showed that the modified bi-quintic B-spline finite element method is remarkably a successful numerical technique. In conclusion, this method with a different choice of B-spline basis functions and element numbers can also be applied to more general non-linear partial differential equations arising in physics and mathematics and resulting in both two dimensional and coupled two dimensional equations.

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