

# Effect of Intermediate Toxic Product on the Survival of a Resource Dependent Species: A Modeling Study

Ram Naresh<sup>1,\*</sup>, Shyam Sundar<sup>2</sup>

<sup>1</sup>Department of Mathematics, H. B. Technological Institute, Kanpur, 208002, India

<sup>2</sup>Department of Mathematics, P.S. Institute of Technology, Bhauti, Kanpur, 208020, India

**Abstract** A nonlinear model is proposed and analysed to study the effect of intermediate toxic product on the survival of a resource dependent species in a polluted environment. It is assumed that when resource biomass uptakes pollutants/toxicants, a liquid (sap) present in the body of biomass reacts with such toxicants and as such intermediate toxic product is formed. This toxic substance then affects the biomass and the species dependent on it. The analysis of the model shows that with increase in the cumulative emission rate of toxicants in the atmosphere, the densities of resource biomass and the species dependent on it decrease and attain their lowest equilibrium. If the rate of emission of toxicants is large enough, the resource biomass may become extinct under certain conditions and the species dependent on it may not survive. The model analysis also suggests that if the formation of intermediate toxic product is restricted by way of controlling the emission of toxicants in the environment, the resulting growth of resource biomass would lead to survival of species dependent on it.

**Keywords** Resource Biomass, Resource Dependent Species, Intermediate Toxic Product, Stability, Numerical Simulation

## 1. Introduction

Various kinds of pollutants/toxicants discharged into the environment very often affect the resources and the species dependent on them[1-14]. In particular, Freedman and Shukla[1] studied the effect of a toxicant on single species and predator-prey systems. They have assumed that the intrinsic growth rate of species decreases with increase in uptaken concentration of toxicant whereas its carrying capacity decreases with the environmental concentration of toxicant. The effect of toxicants on a two species competitive system was studied by Chattopadhyaya[2]. Samanta and Matti[3] studied the effect of toxicant on a single species by considering the three cases of toxicant emission i.e. instantaneous spill, constant emission and rapidly fluctuating random emission of toxicant into the environment. It was shown that the toxicant concentration emitted instantaneously would not be sufficient to kill the population whereas the constant emission makes the population to settle down to steady state. In the third case (rapidly fluctuating random emission), the stability (local) of the system would depend on the washout rate of toxicant from the environment.

In recent years, some mathematical models have also been proposed to study the existence and survival of resource

dependent species living in a polluted environment [15-21]. Dubey and Hussain[15] proposed models for the survival of two competing species dependent on a resource in an industrial environment. They assumed competing species to be partially dependent, wholly dependent or predating on the resource biomass. They concluded that an appropriate level of resource biomass can be maintained to ensure the survival of species if suitable efforts to conserve the resource biomass and to control the undesired label of industrialization pressure are made. Shukla et al.[16] studied a model for survival of resource dependent population to see the effect of toxicant emitted from external sources as well as formed by its precursors. They have shown that the densities of resource and the population decrease as the cumulative emission rate of environmental toxicant increases.

It is pointed out here that in the above studies, the effect of toxicant on resource biomass and species dependent on it is considered but without incorporating the process of formation of intermediate toxic product. However, in real situations the intrinsic growth of resource biomass is, very often affected by intermediate toxic product which is formed inside the resource biomass due to some metabolic reactions. This resource biomass affected by the intermediate toxic product then affects the species dependent on it. In this direction, Naresh et al.[22] presented a mathematical model to study the effect of intermediate toxic product formed by uptake of a toxicant on plant biomass. They have shown that, as the rate of emission of toxicant increases, the equilibrium label of plant biomass decreases. Since the toxicants emitted

\* Corresponding author:

ramntripathi@yahoo.com (Ram Naresh)

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into the atmosphere are uptaken by the resource biomass, an intermediate toxic product is formed which then affects the growth of resource biomass and the species dependent on it. Hence, in the present investigation, our main purpose is to study the effect of intermediate toxic product on the survival of resource dependent species. Thus, we propose a nonlinear mathematical model to see the effect of intermediate toxic product on the resource biomass and on the survival of species dependent on it[22].

The paper is organized as follows, in Section 2 we present the mathematical model. The equilibrium analysis is carried out in Section 3 and in Section 4, the stability analysis of the model is presented. In Section 5, we present the numerical simulations of the model and conclusions are provided in Section 6.

## 2. Mathematical Model

The following assumptions have been made in the modeling process,

1. The densities of resource and species dependent on it are governed by logistic models.
2. The growth rate of resource biomass decreases with increase in the concentration of intermediate toxic product.
3. The growth rate of resource dependent species increases with increase in the density of resource biomass.
4. The carrying capacities of resource biomass as well as that of species dependent on it decrease with environmental concentration of toxicants.

Let  $B(t)$  be the resource biomass density, affected by toxicants emitted at a constant rate  $Q$  in the environment. It is assumed that, when the amount of toxicants uptaken by resource biomass interacts with the bio fluid (sap) present inside the biomass, an intermediate toxic product is formed which affects the growth of resource biomass. Let  $N(t)$  be the density of resource dependent species, its growth rate  $r_2(B)$  is enhanced by the resource biomass density. Let  $T(t)$  be the cumulative concentration of toxicants in the environment with natural depletion rate  $\delta$  and  $\alpha$  is the depletion rate coefficient of toxicants due to uptake by resource biomass. The environmental concentration of toxicants affect the carrying capacities  $K_1(T)$  and  $K_2(T)$  of the resource biomass and the resource dependent species respectively. It is assumed that the uptake of the toxicants by the resource biomass is directly proportional to the density of resource biomass and the concentration of toxicants. Let  $U(t)$  be the concentration of toxicants uptaken by the resource biomass with  $\delta_1$  as natural depletion rate coefficient and  $U_1(t)$  be the concentration of intermediate toxic product formed with a rate  $\alpha_1$  and  $\alpha_0$  as its natural depletion rate coefficient.

Keeping in view of the above assumptions and considerations, the system dynamics is assumed to be governed by the following nonlinear ordinary differential equations,

$$\frac{dB}{dt} = r_1(U_1)B - \frac{r_{10}B^2}{K_1(T)} \quad (1)$$

$$\frac{dN}{dt} = r_2(B)N - \frac{r_{20}N^2}{K_2(T)} \quad (2)$$

$$\frac{dT}{dt} = Q - \delta T - \alpha B T + \pi \nu UB + \theta_1 \delta_1 U \quad (3)$$

$$\frac{dU}{dt} = \alpha B T - \delta_1 U - \nu UB \quad (4)$$

$$\frac{dU_1}{dt} = \alpha_1 U - \alpha_0 U_1 - \nu_1 U_1 B \quad (5)$$

$$N(0) = N_0 > 0, B(0) = B_0 > 0, T(0) \geq 0,$$

$$U(0) \geq 0, U_1(0) > 0$$

The uptaken concentration of toxicants and the concentration of intermediate toxic product are also assumed to be depleted by an amount  $\nu UB$  and  $\nu_1 U_1 B$  respectively due to falling of biomass on the ground. A fraction of the depleted amount  $\nu UB$  (i.e.  $\pi \nu UB$ ) may also re-enter the environment, thus increasing the growth of toxicants. The constants  $0 \leq \pi \leq 1$  and  $0 \leq \theta_1 \leq 1$  are reversible rate coefficients. All the constants are assumed to be non-negative.

In the model, the function  $r_1(U_1)$  denotes the intrinsic growth rate of resource biomass which decreases as the concentration of intermediate toxic product  $U_1$  increases and hence, we assume that,

$$r_1(0) = r_{10} > 0, \quad r_1'(U_1) < 0 \quad \text{for } U_1 > 0$$

The function  $K_1(T)$  denotes the carrying capacity of resource biomass which decreases as the concentration of toxicant  $T$  increases and hence,

$$K_1(0) = K_{10} > 0, \quad K_1'(T) < 0 \quad \text{for } T > 0$$

The function  $r_2(B)$  denotes the intrinsic growth rate of resource dependent species which increases as the resource biomass density  $B$  increases and hence,

$$r_2(0) = r_{20}, \quad r_2'(B) > 0 \quad \text{for } B \geq 0$$

The function  $K_2(T)$  denotes the carrying capacity of resource biomass which decreases as the concentration of toxicant  $T$  increases and hence,

$$K_2(0) = K_{20} > 0, \quad K_2'(T) < 0 \quad \text{for } T > 0$$

## 3. Equilibrium Analysis

The model (1) – (5) has the following four equilibria namely,

1.  $E_0 \left( 0, 0, \frac{Q}{\delta}, 0, 0 \right)$
2.  $E_1 \left( 0, \bar{N}, \bar{T}, 0, 0 \right)$ , where  $\bar{T} = \frac{Q}{\delta}$  and  $\bar{N} = K(\bar{T})$
3.  $E_2 \left( \bar{B}, 0, \bar{T}, \bar{U}, \bar{U}_1 \right)$
4.  $E^* \left( B^*, N^*, T^*, U^*, U_1^* \right)$

The existence of  $E_0$  and  $E_1$  is obvious.

### 3.1. Existence and Uniqueness of $E_2$

The positive solution of variables in equilibrium  $E_2$  is given by the following equations which are obtained by putting the right hand sides of model equations (1) - (5) to zero

$$B = \frac{r_1(U_1)K_1(T)}{r_{10}} \quad (6)$$

$$Q - \delta T - \alpha B T + \pi \nu U B + \theta_1 \delta_1 U = 0 \quad (7)$$

$$\alpha B T - \delta_1 U - \nu U B = 0 \quad (8)$$

$$\alpha_1 U - \alpha_0 U_1 - \nu_1 U_1 B = 0 \quad (9)$$

From eqs. (7) and (8) we have,

$$T = \frac{Q(\delta_1 + \nu B)}{\delta(\delta_1 + \nu B) + (1 - \theta_1)\alpha\delta_1 B + (1 - \pi)\alpha\nu B^2} = f(B) \quad (10)$$

From eq. (8) we get,

$$U = \frac{\alpha B f(B)}{\delta_1 + \nu B} = g(B) \quad (11)$$

From eq. (9) we get,

$$U_1 = \frac{\alpha_1 U}{\alpha_0 + \nu_1 B} = \frac{\alpha_1 g(B)}{\alpha_0 + \nu_1 B} = h(B) \quad (12)$$

Using eq. (6) we assume that,

$$F(B) = r_{10}B - r_1(U_1)K_1(T)$$

which gives

$$F(B) = r_{10}B - r_1\{h(B)\}K_1\{f(B)\} \quad (13)$$

From eq. (13), it can be seen that

$$F(0) = -r_1(0)K_1(f(0)) < 0 \quad (14)$$

and

$$F(K_{10}) = r_{10}K_{10} - [r_1\{h(B)\}]_{B=K_{10}} \quad (15)$$

$$x[K_1\{f(B)\}]_{B=K_{10}} > 0$$

also

$$F'(B) = r_{10} - [r_1 K_1'(T) f'(B) + K_1 r_1'(U_1) h'(B)] \quad (16)$$

From eqs. (14) and (15) it is clear that there exist a root  $\tilde{B}$  of  $F(B) = 0$  in  $0 \leq B \leq K_{10}$ . The root will be unique, provided,

$$r_{10} - [r_1 K_1'(T) f'(B) + K_1 r_1'(U_1) h'(B)] > 0$$

where,

$$h'(B) = \frac{\alpha_1}{(\alpha_0 + \nu_1 B)^2} [(\alpha_0 + \nu_1 B)g'(B) - \nu_1 g(B)]$$

$$g'(B) = \frac{\alpha_1}{(\delta_1 + \nu B)^2} [(\delta_1 + \nu B)Bf'(B) + \delta_1 f(B)]$$

$$f'(B) = -\frac{Q[\delta_1^2(1 - \theta_1)\alpha + (1 - \pi)\alpha\nu^2 B^2 + 2(1 - \pi)\delta_1\alpha\nu B]}{[\delta(\delta_1 + \nu B) + (1 - \theta_1)\delta_1\alpha B + (1 - \pi)\alpha\nu B^2]} < 0$$

Knowing the value of  $\tilde{B}$ , the values of  $\tilde{T}$ ,  $\tilde{U}$  and  $\tilde{U}_1$  can be found from eqs. (10), (11) and (12) respectively.

### 3.2. Existence and Uniqueness of $E^*$

The positive solution of variable in  $E^*$  is given by the

following algebraic equations,

$$B = \frac{r_1(U_1)K_1(T)}{r_{10}} \quad (17)$$

$$N = \frac{r_2(B)K_2(T)}{r_{20}} \quad (18)$$

$$T = \frac{Q(\delta_1 + \nu B)}{\delta(\delta_1 + \nu B) + (1 - \theta_1)\alpha\delta_1 B + (1 - \pi)\alpha\nu B^2} = f(B) \quad (19)$$

$$U = \frac{\alpha B f(B)}{\delta_1 + \nu B} = g(B) \quad (20)$$

$$U_1 = \frac{\alpha_1 U}{\alpha_0 + \nu_1 B} = \frac{\alpha_1 g(B)}{\alpha_0 + \nu_1 B} = h(B) \quad (21)$$

As before, let  $F(B) = r_{10}B - r_1\{h(B)\}K_1\{f(B)\}$

From this, it can be seen that  $F(0) < 0$  and  $F(K_{10}) > 0$

showing the existence of a root  $B^*$  of  $F(B) = 0$  in  $0 \leq B \leq K_{10}$  and the root will be unique, provided,

$$r_{10} - [r_1 K_1'(T) f'(B) + K_1 r_1'(U_1) h'(B)] > 0$$

Knowing the value of  $B^*$ , we can find the values of  $N^*$ ,  $T^*$ ,  $U^*$  and  $U_1^*$  from eqs. (18), (19), (20) and (21) respectively.

In the following we analyse the stability behavior of above equilibria.

## 4. Stability Analysis

In this section, we describe stability analysis of different equilibria.

### Theorem 1

(i) Equilibria  $E_0\left(0, 0, \frac{Q}{\delta}, 0, 0\right)$ ,  $E_1(0, \bar{N}, \bar{T}, 0, 0)$  and

$E_2(\tilde{B}, 0, \tilde{T}, \tilde{U}, \tilde{U}_1)$  are unstable.

(ii) If the following inequalities hold,

$$\left[ (\alpha T^* + \pi \nu U^*) + \frac{r_{10} B^{*2}}{[K_1(T)]^2} \{-K_1'(T^*)\} \right]^2 < \frac{1}{3} \frac{r_{10} B^*}{K_1(T^*)} (\delta + \alpha B^*) \quad (22)$$

$$[-r_1'(U_1^*)] B^* + \nu_1 U_1^*]^2 < \frac{1}{2} \frac{r_{10} B^*}{K_1(T^*)} (\alpha_0 + \nu_1 B^*) \quad (23)$$

$$[(\pi \nu B^* + \theta_1 \delta_1) + k_3 \alpha B^*]^2 < \frac{4}{9} k_3 (\delta + \alpha B^*) (\delta_1 + \nu B^*) \quad (24)$$

$$\alpha_1^2 < \frac{2}{9} \frac{r_{10} B^* (\delta_1 + \nu B^*)^2 (\alpha_0 + \nu_1 B^*)}{K_1(T^*) (\alpha T^* + \nu U^*)^2} \quad (25)$$

where  $k_3 < \frac{1}{3} \frac{r_{10} B^* (\delta_1 + \nu B^*)}{K_1(T^*) (\alpha T^* + \nu U^*)^2}$

then  $E^*$  is locally asymptotically stable (See Appendix-A for proof).

To establish the nonlinear asymptotic stability of  $E^*$ , we need the bounds of different variables. For this we propose

the following region of attraction, stated without proof Freedman and So[23].

**Lemma 1** The set

$$\Omega = \left\{ (B, N, T, U, U_1) : 0 \leq B \leq K_{10}, 0 \leq N \leq N_c, \right. \\ \left. 0 \leq T + U \leq \frac{Q}{\delta_m}, 0 \leq U_1 \leq \frac{\alpha_1 Q}{\alpha_0 \delta_m} \right\}$$

attracts all solutions initiating in the interior of non-negative octant, where  $N_c = \frac{r_2(K_{10})K_2(f(K_{10}))}{r_{20}}$  and

$$\delta_m = \min(\delta, \delta_1(1 - \theta_1)).$$

**Theorem 2.**

Let  $K_{1m} \leq K_1(T) \leq K_{10}$ ,  $K_{2m} \leq K_2(T) \leq K_{20}$ ,  $0 \leq -r'_1(U_1) \leq p$ ,  $0 \leq r'_2(B) \leq r$ ,  $0 \leq -K'_1(T) \leq q$ ,  $0 \leq K'_2(T) \leq s$ ,  $|\xi_1(U_1)| \leq p$ ,  $|\xi_2(B)| \leq r$ ,  $|\eta_1(T)| \leq \frac{q}{K_{1m}^2}$ ,  $|\eta_2(T)| \leq \frac{s}{K_{2m}^2}$

satisfying in  $\Omega$  for some constants  $K_{1m}, K_{2m}, K_{10}, K_{20}, p, q, r, s$ , then if the following inequalities hold in  $\Omega$ ,

$$\left[ r_{10} K_{10} \frac{q}{K_{1m}^2} + (\alpha + \pi \nu) \frac{Q}{\delta_m} \right]^2 < \frac{1}{3} \frac{r_{10}}{K_1(T^*)} (\delta + \alpha B^*) \quad (26)$$

$$\left[ p + \nu_1 \frac{Q \alpha_1}{\delta_m \alpha_0} \right]^2 < \frac{1}{2} \frac{r_{10}}{K_1(T^*)} (\alpha_0 + \nu_1 B^*) \quad (27)$$

$$[(\pi \nu B^* + \theta_1 \delta_1) + m_3 \alpha B^*]^2 < \frac{4}{9} m_3 (\delta + \alpha B^*) (\delta_1 + \nu B^*) \quad (28)$$

$$\alpha_1^2 < \frac{2}{9} \frac{r_{10} (\delta_1 + \nu B^*)^2 (\alpha_0 + \nu_1 B^*)}{K_1(T^*) (\alpha + \nu)^2 \left( \frac{Q}{\delta_m} \right)^2} \quad (29)$$

$$\text{where } m_3 < \frac{1}{3} \frac{r_{10} (\delta_1 + \nu B^*)}{K_1(T^*) (\alpha + \nu)^2 \left( \frac{Q}{\delta_m} \right)^2}$$

$E^*$  is nonlinearly asymptotically stable with respect to all solutions initiating in the interior of the first octant. (See Appendix-B for proof)

**Remarks:**

1. If  $Q, \alpha, \alpha_1, \pi$  and  $\delta_1$  are very small, then the possibility of satisfying conditions (22) – (29) is more plausible showing that these parameters have destabilizing effect on the system.

2. If  $\alpha = 0, \alpha_1 = 0, \pi = 0$  and  $\delta_1 = 0$ , then the conditions (24), (25), (28) and (29) are satisfied automatically.

The above analysis imply that as the rate of introduction of toxicants in the environment increases, then under certain conditions the densities of resource biomass and resource dependent species decrease and settle down to their respective equilibrium levels. It is pointed here that the magnitude of equilibrium of species would mainly depend upon the resource biomass density affected by intermediate toxic product formed inside the biomass due to some metabolic changes. The density of resource biomass

decreases as cumulative concentration of toxicants in the environment increases and it may even tend to zero for very high concentration of toxicants and then the species dependent on it may not survive.

## 5. Numerical Simulations

In this section, we analyse the model (1) – (5) numerically with the help of MAPLE 7.0 to study the behaviour of the system for different values of parameters. For this we assume that,

$$r(U_1) = r_{10} - a_1 U_1, \quad K_1(T) = K_{10} - b_1 T$$

$$r(B) = r_{20} + a_2 B, \quad K_2(T) = K_{20} - b_2 T$$

Now we consider the following set of parameter values,

$$Q = 5, a_1 = 0.01, r_{10} = 12, b_1 = 0.3, m_1 = 1.5,$$

$$\pi = 0.0015, \delta = 0.9, \alpha = 0.6, \alpha_1 = 0.01, \nu = 0.65$$

$$\alpha_0 = 0.05, K_{10} = 6.75, r_{20} = 10, a_2 = 0.07,$$

$$b_2 = 0.2, K_{20} = 8, \theta_1 = 0.2, \nu_1 = 0.1$$

Equilibrium values of different variable in  $E^*$  are obtained as,

$$B^* = 6.4344695, N^* = 8.1405519, T^* = 1.0515293,$$

$$U^* = 0.9263452, U_1^* = 0.0133585$$

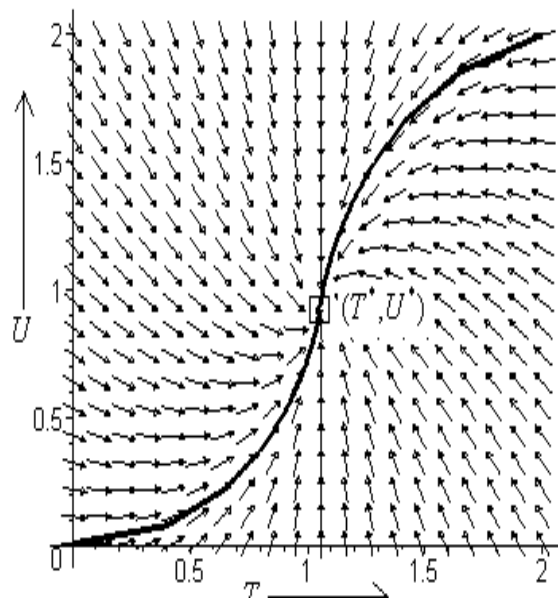
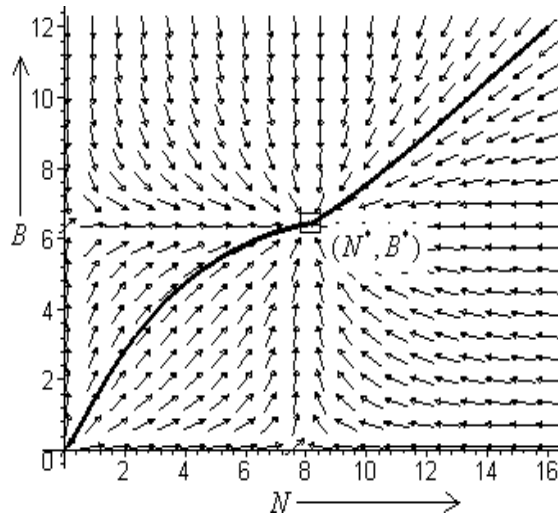
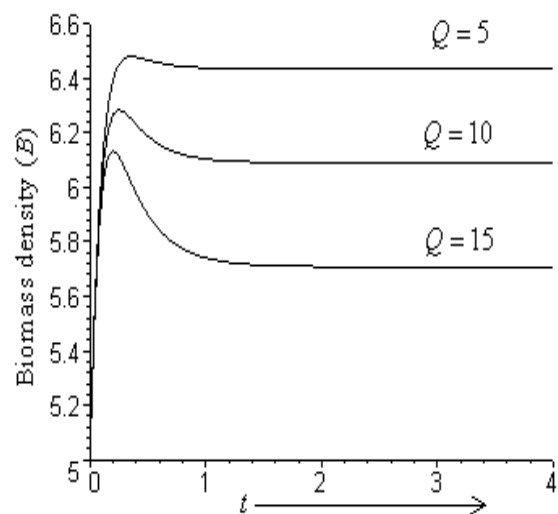
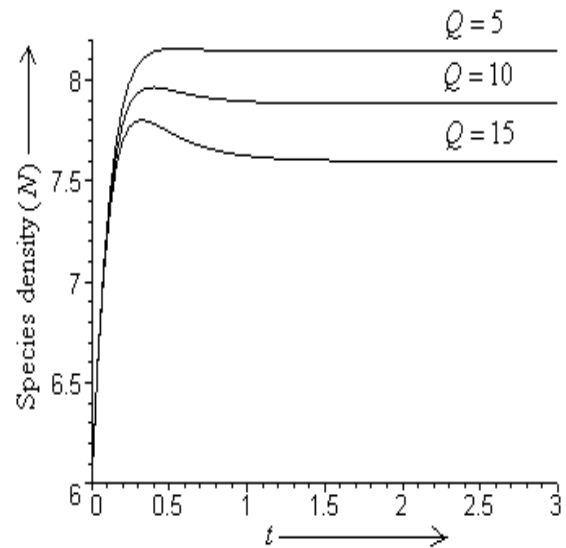
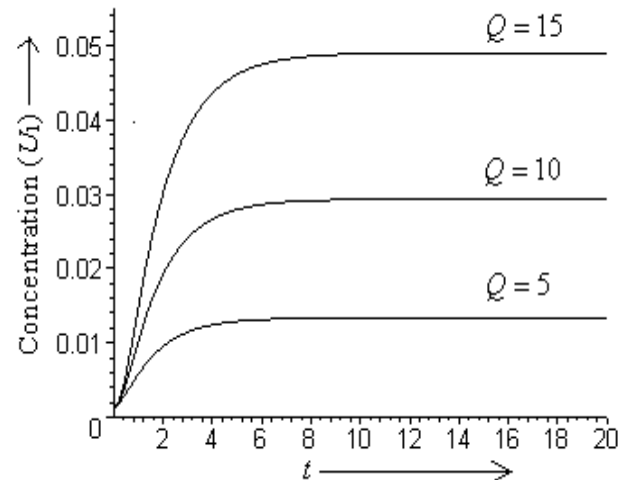
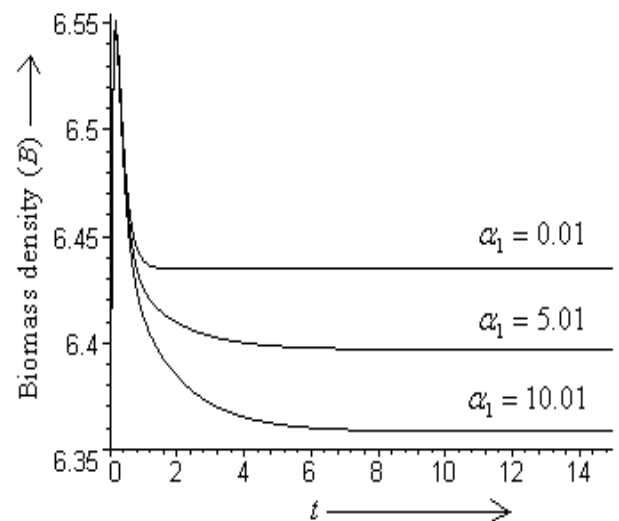
Eigen values of the matrix corresponding to  $E^*$  are given by,

$$-10.45041287, -12.30080454, -4.580458550,$$

$$-4.261707507, -0.6934297045$$

Since all the eigen values corresponding to  $E^*$  are negative and therefore  $E^*$  is locally asymptotically stable.

The nonlinear asymptotic stability behaviour of  $E^*$  in  $T-U$  and  $N-B$  plane is shown in the figure 1 and figure 2 respectively. In these figures, it is shown that the trajectories started at any point in the region approaches the equilibrium point  $E^*$  showing that the equilibrium is nonlinearly asymptotically stable. In figures 3, 4, 5, the variation of the densities of biomass ( $B$ ), resource dependent species ( $N$ ) and the concentration of intermediate toxic product ( $U_1$ ) is shown with time at different values of rate of emission of toxicants in the environment i.e. at  $Q = 5, 10, 15$  respectively. From these figures, it is observed that the densities of biomass and species dependent on it decrease as the rate of introduction of toxicants increases and settle down to their respective equilibrium levels, while the concentration of intermediate toxic product increases. In figures 6 and 7, the variation of the densities of biomass ( $B$ ) and resource dependent species ( $N$ ) is shown with time at different values of rate of formation of intermediate toxic product ( $U_1$ ) i.e. at  $\alpha_1 = 0.01, 5.01, 10.01$  respectively. From these figures, it can be seen that the densities of biomass and species dependent on it decrease as the rate of formation of intermediate toxic product increases and settle down to their respective equilibrium levels. In figure 8, it is shown that if the growth of resource biomass increases, the density of species dependent on it also increases.

Figure 1. Nonlinear stability in  $T-U$  planeFigure 2. Nonlinear stability in  $N-B$  planeFigure 3. Variation of biomass density  $B$  with time ' $t$ ' for different values of  $Q$ Figure 4. Variation of density of resource dependent species  $N$  with time ' $t$ ' for different values of  $Q$ Figure 5. Variation of concentration of intermediate toxic product  $U_1$  with time ' $t$ ' for different values of  $Q$ Figure 6. Variation of biomass density  $B$  with time ' $t$ ' for different values of  $\alpha_1$

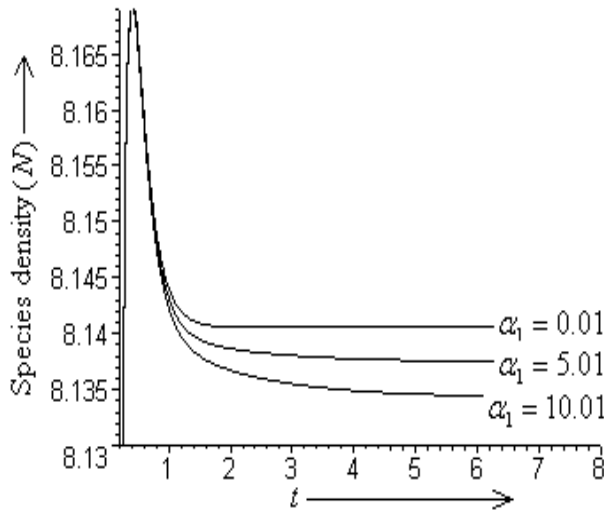


Figure 7. Variation of density of resource dependent species  $N$  with time ' $t$ ' for different values of  $\alpha_1$

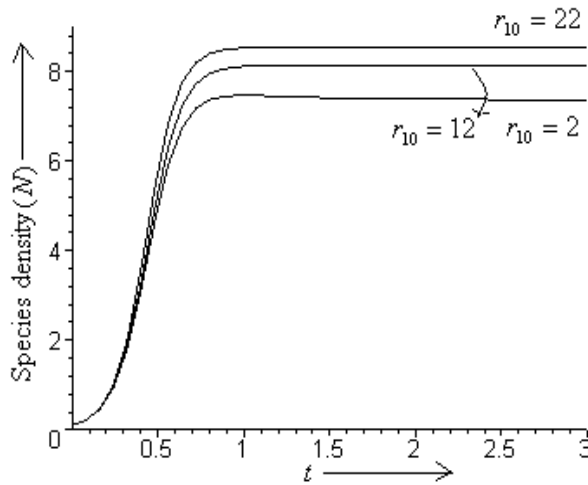


Figure 8. Variation of density of resource dependent species  $N$  with time ' $t$ ' for different values of  $r_{10}$

## 6. Conclusions

In this paper, we have proposed a nonlinear mathematical model to study the effect of intermediate toxic product on the survival of a resource dependent species living in a polluted environment. It is assumed that the growth of resource biomass is affected by the intermediate toxic product formed inside the resource biomass due to some metabolic reactions when toxicants present in the environment are uptaken by resource biomass. This affected resource biomass by the intermediate toxic product then affects the resource dependent species. The model is analysed using stability theory of differential equations and computer simulations. It is shown that densities of resource biomass and species dependent on it decrease as the rate of introduction of toxicants increases in the environment and settle down to their respective equilibrium levels while the concentration of intermediate toxic product increases. Further, as the rate of formation of intermediate toxic product increases, the

densities of resource biomass and the species dependent on it also decrease and their magnitudes are less than their respective densities when they are not affected by toxicant. If the rate of emission of toxicants in the environment is large enough, then under certain conditions the resource biomass may become extinct due to increased level of intermediate toxic product affecting the growth of biomass and as such the species dependent on it may not survive. It is also observed that the density of species dependent on resource biomass increases if the growth of resource biomass increases. Thus, if the formation of intermediate toxic product is restricted by way of controlling the emission of toxicants in the environment, the resulting growth of resource biomass would lead to survival of species dependent on it.

## APPENDIX-A

### Proof of the Theorem 1.

(i) The variational matrix corresponding to  $E_0(0, 0, \frac{Q}{\delta}, 0, 0)$  is given by,

$$M_0 = \begin{bmatrix} r_{01} & 0 & 0 & 0 & 0 \\ 0 & r_{20} & 0 & 0 & 0 \\ -\alpha \frac{Q}{\delta} & 0 & -\delta & \theta_1 \delta_1 & 0 \\ \alpha \frac{Q}{\delta} & 0 & 0 & -\delta_1 & 0 \\ 0 & 0 & 0 & \alpha_1 & -\alpha_0 \end{bmatrix}$$

It can be seen that the two eigen values  $r_{10}, r_{20}$  of  $M_0$  are positive, therefore  $E_0$  is a saddle point.

The variational matrix corresponding to  $E_1(0, \bar{N}, \bar{T}, 0, 0)$  is given by,

$$M_1 = \begin{bmatrix} r_{10} & 0 & 0 & 0 & 0 \\ r'_2(0)\bar{N} & r_2(0) - \frac{2r_{20}\bar{N}}{K_2(\bar{T})} & \frac{r_{20}\bar{N}^2}{[K_2(\bar{T})]^2} K'_2(\bar{T}) & 0 & 0 \\ -\alpha \bar{T} & 0 & -\delta & \theta_1 \delta_1 & 0 \\ \alpha \bar{T} & 0 & 0 & -\delta_1 & 0 \\ 0 & 0 & 0 & \alpha_1 & -\alpha_0 \end{bmatrix}$$

From which we note that  $E_1(0, \bar{N}, \bar{T}, 0, 0)$  is a saddle point.

Similarly, it can also be checked that equilibrium point  $E_2(\bar{B}, 0, \bar{T}, \bar{U}, \bar{U}_1)$  is unstable in  $N$ -direction.

(ii) Consider the following positive definite function about  $E^*$ ,

$$V = \frac{1}{2} [k_0 b^2 + k_1 n^2 + k_2 \tau^2 + k_3 u^2 + k_4 u_1^2] \quad (A1)$$

where,

$$B = B^* + b, N = N^* + n, T = T^* + \tau, U = U^* + u, U_1 = U_1^* + u_1$$

Differentiating (A1) with respect to ' $t$ ' we get

$$\dot{V} = k_0 b \dot{b} + k_1 n \dot{n} + k_2 \tau \dot{\tau} + k_3 u \dot{u} + k_4 u_1 \dot{u}_1 \quad (A2)$$

Now using the linearized system of the model (1) – (5) corresponding to  $E^*(B^*, N^*, T^*, U^*, U_1^*)$  in (A2), we have,

$$\begin{aligned}\dot{V} = & -k_0 \frac{r_{10} B^*}{K_1(T^*)} b^2 - k_1 \frac{r_{20} N^*}{K_2(T^*)} n^2 - k_2 (\alpha B^* + \delta) \tau^2 \\ & - k_3 (\delta_1 + \nu B^*) u^2 - k_4 (\alpha_0 + \nu_1 B^*) u_1^2 \\ & + k_2 r_2' (B^*) N^* b n \\ & + \left[ k_0 \frac{r_{10} B^{*2}}{[K_1(T)]^2} K_1'(T^*) - k_2 (\alpha T^* - \pi \nu U^*) \right] b \tau \\ & + k_3 (\alpha T^* - \nu U^*) b u + [k_0 r_1' (U_1^*) B^* - k_4 \nu_1 U_1^*] b u_1 \\ & + k_1 \frac{r_{20} N^{*2}}{[K_2(T)]^2} K_2'(T^*) n \tau \\ & + [k_2 (\pi \nu B^* + \theta_1 \delta_1) + k_3 \alpha B^*] \tau u + k_4 \alpha_1 u u_1\end{aligned}$$

Now  $\dot{V}$  will be negative definite, if the following inequalities are satisfied,

$$k_1 [r_2' (B^*) N^*]^2 < \frac{1}{2} k_0 \frac{r_{10} B^*}{K_1(T^*)} \frac{r_{20} N^*}{K_2(T^*)} \quad (A3)$$

$$\begin{aligned}& \left[ k_0 \frac{r_{10} B^{*2}}{[K_1(T)]^2} K_1'(T^*) - k_2 (\alpha T^* - \pi \nu U^*) \right]^2 \\ & < \frac{1}{3} k_0 k_2 \frac{r_{10} B^*}{K_1(T^*)} (\alpha B^* + \delta)\end{aligned} \quad (A4)$$

$$[k_3 (\alpha T^* - \nu U^*)]^2 < \frac{1}{3} k_0 k_3 \frac{r_{10} B^*}{K_1(T^*)} (\delta_1 + \nu B^*) \quad (A5)$$

$$[k_0 r_1' (U_1^*) B^* - k_4 \nu_1 U_1^*]^2 < \frac{1}{2} k_0 k_4 \frac{r_{10} B^*}{K_1(T^*)} (\alpha_0 + \nu_1 B^*) \quad (A6)$$

$$\left[ k_1 \frac{r_{20} N^{*2}}{[K_2(T)]^2} K_2'(T^*) \right]^2 < \frac{2}{3} k_1 k_2 \frac{r_{20} N^*}{K_2(T^*)} (\alpha B^* + \delta) \quad (A7)$$

$$[k_2 (\pi \nu B^* + \theta_1 \delta_1) + k_3 \alpha B^*]^2 < \frac{4}{9} k_2 k_3 (\alpha B^* + \delta) (\delta_1 + \nu B^*) \quad (A8)$$

$$[k_4 \alpha_1]^2 < \frac{2}{3} k_3 k_4 (\delta_1 + \nu B^*) (\alpha_0 + \nu_1 B^*) \quad (A9)$$

Under the assumptions  $k_0 = k_2 = k_4 = 1$ ,

$$\begin{aligned}k_1 & < \min \left\{ \frac{1}{2} \frac{r_{10} B^*}{K_1(T^*)} \frac{r_{20}}{K_2(T^*)} \frac{1}{[r_2' (B^*)]^2 N^*}, \right. \\ & \left. \frac{2}{3} (\delta + \alpha B^*) \frac{K_2'^3 (T^*)}{r_{20} N^{*3} K_2'^2 (T^*)} \right\} \\ k_3 & < \frac{1}{3} \frac{r_{10} B^* (\delta_1 + \nu B^*)}{K_1(T^*) (\alpha T^* + \nu U^*)^2}\end{aligned}$$

$\dot{V}$  will be negative definite provided the conditions (22) – (25) are satisfied, showing that  $V$  is a Liapunov function and hence  $E^*$  is locally asymptotically stable.

## APPENDIX-B

### Proof of the Theorem 2.

Consider the following positive definite function about  $E^*$ ,

$$U = m_0 \left[ B - B^* - B^* \log \frac{B}{B^*} \right] + m_1 \left[ N - N^* - N^* \log \frac{N}{N^*} \right] \quad (B1) \\ + \frac{1}{2} m_2 (T - T^*)^2 + \frac{1}{2} m_3 [U - U^*]^2 + \frac{1}{2} m_4 [U_1 - U_1^*]^2$$

where the constants  $m_i$  ( $i = 0, 1, 2, 3, 4$ ) can be chosen appropriately.

On differentiation we get,

$$\begin{aligned}\dot{U} = & m_0 (B - B^*) \frac{1}{B} \frac{dB}{dt} + m_1 (N - N^*) \frac{1}{N} \frac{dN}{dt} + m_2 [T - T^*] \frac{dT}{dt} \\ & + m_3 [U - U^*] \frac{dU}{dt} + m_4 [U_1 - U_1^*] \frac{dU_1}{dt} \\ & + m_0 (B - B^*) \left[ \frac{[U_1 - U_1^*] \xi_1 (U_1) - r_{10} B \eta_1 (T) (T - T^*)}{K_1(T^*)} \right] \\ & + m_1 (N - N^*) \left[ \frac{[B - B^*] \xi_2 (B) - r_{20} N \eta_2 (T) (T - T^*)}{K_2(T^*)} \right] \\ & + m_2 (T - T^*) \left[ \frac{-(\delta + \alpha B^*) [T - T^*] - (\alpha T - \pi \nu U) (B - B^*)}{(\theta_1 \delta_1 + \pi \nu B^*) (U - U^*)} \right] \\ & + m_3 (U - U^*) \left[ \frac{\alpha T - \nu U (B - B^*) + \alpha B^* [T - T^*]}{-(\delta_1 + \nu B^*) (U - U^*)} \right] \\ & + m_4 (U_1 - U_1^*) \left[ \frac{\alpha_1 (U - U^*) - (\alpha_0 + \nu_1 B^*) (U_1 - U_1^*)}{-\nu_1 U_1 (B - B^*)} \right]\end{aligned}$$

where

$$\begin{aligned}\xi_1 (U_1) & = \begin{cases} \frac{r_1 (U_1) - r_1 (U_1^*)}{U_1 - U_1^*}, & U_1 \neq U_1^* \\ r_1' (U_1), & U_1 = U_1^* \end{cases} \\ \eta_1 (T) & = \begin{cases} \frac{1}{K_1(T)} - \frac{1}{K_1(T^*)}, & T \neq T^* \\ -\frac{K_1'(T)}{K_1^2(T)}, & T = T^* \end{cases} \\ \xi_2 (B) & = \begin{cases} \frac{r_2 (B) - r_2 (B^*)}{B - B^*}, & B \neq B^* \\ r_2' (B), & B = B^* \end{cases} \\ \eta_2 (T) & = \begin{cases} \frac{1}{K_2(T)} - \frac{1}{K_2(T^*)}, & T \neq T^* \\ -\frac{K_2'(T)}{K_2^2(T)}, & T = T^* \end{cases}\end{aligned}$$

$$\begin{aligned}
\dot{U} = & -m_0 \frac{r_{10}}{K_1(T^*)} (B - B^*)^2 - m_1 \frac{r_{20}}{K_2(T^*)} (N - N^*)^2 \\
& - m_2 (\delta + \alpha B^*) [T - T^*]^2 \\
& - m_3 (\delta_1 + \nu B^*) (U - U^*)^2 - m_4 (\alpha_0 + \nu_1 B^*) (U_1 - U_1^*)^2 \\
& + m_1 \xi_2 (B) [B - B^*] (N - N^*) \\
& - [m_1 r_{10} B \eta_1(T) + m_2 (\alpha T - \pi \nu U)] (B - B^*) (T - T^*) \\
& + m_3 (\alpha T - \nu U) (B - B^*) (U - U^*) \\
& + [m_0 \xi_1 (U_1) - m_4 \nu_1 U_1] (B - B^*) (U_1 - U_1^*) \\
& - m_1 r_{20} N \eta_2(T) (N - N^*) (T - T^*) \\
& + [m_2 (\theta_1 \delta_1 + \pi \nu B^*) + m_3 \alpha B^*] (T - T^*) (U - U^*) \\
& + m_4 \alpha_1 (U - U^*) (U_1 - U_1^*)
\end{aligned}$$

Now  $\dot{U}$  will be negative definite under the following sufficient conditions,

$$m_1 [\xi_2(B^*)]^2 < \frac{1}{2} m_0 \frac{r_{10}}{K_1(T^*)} \frac{r_{20}}{K_2(T^*)} \quad (B2)$$

$$[m_0 r_{10} B \eta_1(T) + m_2 (\alpha T - \pi \nu U)]^2 < \frac{1}{3} m_0 m_2 \frac{r_{10}}{K_1(T^*)} (\delta + \alpha B^*) \quad (B3)$$

$$[m_3 (\alpha T - \nu U)]^2 < \frac{1}{3} m_0 m_3 \frac{r_{10}}{K_1(T^*)} (\delta_1 + \nu B^*) \quad (B4)$$

$$[m_0 \xi_1(U_1) - m_4 \nu_1 U_1]^2 < \frac{1}{2} m_0 m_4 \frac{r_{10}}{K_1(T^*)} (\alpha_0 + \nu_1 B^*) \quad (B5)$$

$$[m_1 r_{20} N \eta_2(T)]^2 < \frac{2}{3} m_1 m_2 \frac{r_{20}}{K_2(T^*)} (\delta + \alpha B^*) \quad (B6)$$

$$[m_2 (\pi \nu B^* + \theta_1 \delta_1) + m_3 \alpha B^*]^2 < \frac{4}{9} m_2 m_3 (\delta + \alpha B^*) (\delta_1 + \nu B^*) \quad (B7)$$

$$[m_4 \alpha_1]^2 < \frac{2}{3} m_3 m_4 (\delta_1 + \nu B^*) (\alpha_0 + \nu_1 B^*) \quad (B8)$$

Maximizing LHS and minimizing RHS and choosing  $m_0 = m_2 = m_4 = 1$ ,

$$\begin{aligned}
m_1 & < \min \left\{ \frac{1}{2} \frac{r_{10}}{K_1(T^*)} \frac{r_{20}}{K_2(T^*)} \frac{1}{r^2}, \frac{2}{3} \frac{(\delta + \alpha B^*)}{K_2(T^*) \left( r_{20} N_c \frac{s}{K_{2m}^2} \right)^2} \right\} \\
m_3 & < \frac{1}{3} \frac{r_{10} (\delta_1 + \nu B^*)}{K_1(T^*) (\alpha + \nu)^2 \left( \frac{Q}{\delta_m} \right)^2}
\end{aligned}$$

$\dot{U}$  will be negative definite provided the conditions (26) – (29) are satisfied and hence the theorem.

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