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A Generalization and Study of New Mock Theta Functions

Bhaskar Srivastava

Department of Mathematics and Astronomy, Lucknow University, Lucknow, India

Abstract Two sets of mock theta functions were developed, one by Andrews and the other by Bringmann et al. We have given two generalizations and shown they belong to the class of F_q -functions. Relations between these generalized functions is established. Later we give q-Integral representation and multibasic expansions of these generalized F_q -functions.

Keywords Mock theta function, q-Multibasic expansion and q-Integral

1. Brief History of Mock Theta Functions

Ramanujan in his last letter to Hardy, dated January 12, 1920, to be specific [18, pp. 354-355] gave a list of seventeen functions which he called "mock theta functions". The functions are of a complex variable q defined by a q-series convergent for $|q| < 1$. As q approaches a root of unity. Ramanujan stated, they have certain asymptotic properties, similar to the properties of theta functions, but he conjectured that they are not theta functions. He also stated some identities relating some of the functions to each other. The list was divided into four group of functions of order three, five, five and seven. Watson[23] studied the third order mock theta functions and introduced three new one. Watson proved that the third order functions have asymptotic properties, as stated by Ramanujan and also that they are not theta functions. Watson proved the asymptotic formula, for the fifth order mock theta functions and Selberg for the seventh order mock theta functions, but neither author proved that the functions are not theta functions.

In 1976, Andrews discovered "Lost" Notebook while visiting Trinity College, Cambridge in the mathematical library of the college, written by Ramanujan towards the end of his life. In the "lost" notebook were six more mock theta functions and linear relations between them, Andrews and Hickerson [6] called them of sixth order. On the page 9 of the "Lost" Notebook appears four more mock theta functions, which were called by Choi [9] of tenth order.

Gordon and McIntosh, listed eight functions and called them of eighth order. Later in their survey paper [12] called only four functions of eighth order, the other four were of lower order. Hikami [13] in his work on Mathematical Physics Quantum Invariant of three manifold came across a mock theta function and called it of second order.

Recently in his path breaking paper [5] while studying the q-orthogonal polynomials found some new mock theta functions. The following two mock theta functions are interesting

$$\bar{\psi}_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q)_{2n}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+n}(1-q^{6n+3})\sum_{j=-n}^{n}(-1)^j q^{-j^2},$$

$$\bar{\psi}_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(-q;q)_{2n+1}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+3n}(1-q^{2n+1})\sum_{j=-n}^{n}(-1)^j q^{-j^2},$$

$$\bar{\psi}_2(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}(q;q^2)_n}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}q^{4n^2+2n}(1-q^{4n+2})\sum_{j=-n}^{n}(-1)^j q^{-j^2},$$

and

$$\bar{\psi}_3(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q)_n^2}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2+n}(1-q^{6n+6})\sum_{j=0}^{n}q^{-\binom{j+1}{2}}.$$

Bringmann, Hikami and Lovejoy [8] also found two more new mock theta functions

$$\bar{\phi}_0(q)=\sum_{n=0}^{\infty}q^n(-q)_{2n+1}$$

and

$$\bar{\phi}_1(q)=\sum_{n=0}^{\infty}q^n(-q)_{2n}.$$

In this paper we have given two generalizations of these mock theta functions and have shown they belong to the family of F_q -functions. Being F_q -functions they have unified properties, for example:

(i) The inverse operator $D_{q,x}^{-1}$ of q-differentiation is related to q-integration as

$$D_{q,x}^{-1}f(x)=(1-q)^{-1}\int f(x)d_q(x)$$

(ii) $D_{q,z}^{n}F(z,\alpha)=F(z,\alpha+n)$, where n is a non-negative integer.

The scheme of the paper is as follows:

* Corresponding author:
bhaskarsrivastav@yahoo.com (Bhaskar Srivastava)
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We define a F_q-function in section 3.

The definition of one generalization of these mock theta functions is in section 4.

Section 5 contains relations between these generalized functions.

In section 6 we define a second generalization of these functions and show they are F_q-functions.

Section 7 contains more relations between these generalized functions.

In section 8 the generalized F_q-functions are represented as q-Integrals.

In section 9, multibasic expansions are given for these generalized F_q-functions.

2. Basic Facts

We shall use the following usual basic hypergeometric notations :

For $|q^k| < 1$,

$$(a;q^k)_n = (1-a)(1-aq^k)...(1-aq^{k(n-1)}), \quad n \geq 1$$

$$(a;q^k)_0 = 1,$$

$$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj}).$$

For convenience we shall write

$$(a_1,a_2,......,a_m;q^k)_n = (a_1;q^k)_n(a_2;q^k)_n......(a_m;q^k)_n.$$

When $k=1$, we usually write $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

$$\phi\begin{bmatrix} a_1,...,a_r : c_{1,1},...,c_{1,r_1} : ... : c_{m,1},...c_{m,r_m} \\ b_1,...,b_s : e_{11},...,e_{1,s_1} : ... : e_{m,1},...e_{m,s_m} \end{bmatrix}; q, q_1, ..., q_m; z$$

$$= \sum_{n=0}^{\infty}\frac{(a_1......,a_r;q)_n}{(q,b_1,.......,b_s;q)_n}z^n\left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{1+s-r}\prod_{j=1}^{m}\frac{(c_{j,1},........,c_{j,r_j};q_j)_n}{(e_{j,1},......,e_{j,s_j};q_j)_n}\left[(-1)^n q_j^{\frac{n^2-n}{2}}\right]^{s_j-r_j}.$$

A generalized basic hypergeometric series with base q_1 is defined as

$$_A\varphi_{A-1}\left[a_1,a_2,...,a_A;b_1,b_2...,b_{A-1};q_1,z\right]$$
$$= \sum_{n=0}^{\infty}\frac{(a_1;q_1)_n....(a_A;q_1)_n z^n}{(b_1;q_1)_n....(b_{A-1};q_1)_n(q_1;q_1)_n}, \quad |z| < 1.$$

3. Definition of F_q-Functions

Truesdell [22] in his book, "Unified theory of special functions" calls a function F-function, if it satisfies the functional equation

$$\frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1).$$

The q-analogue is: A function is called F_q-Function if it satisfies the functional equation

$$D_{q,z}\ F(z,\alpha) = F(z,\alpha+1),$$

where

$$zD_{q,z}\ F(z,\alpha) = F(z,\alpha) - F(zq,\alpha).$$

4. Generalization of Mock Theta Functions and are F_q-Functions

We give a generalization of these mock theta functions and show they are F_q-functions.

Definition of the generalized functions:

$$\bar{\psi}_0(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}}, \tag{4.1}$$

$$\bar{\psi}_1(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+n+n\alpha}}{(-q;q)_{2n+1}}, \tag{4.2}$$

$$\bar{\psi}_2(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+n+n\alpha}(q;q^2)_n}{(q^2;q^2)_n(-q;q)_{2n}}, \tag{4.3}$$

$$\bar{\psi}_3(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-n+n\alpha}(-q;q)_n}{(q;q)_n(q^{1/2};q)_n(-q^{1/2};q)_n}, \tag{4.4}$$

$$\bar{\phi}_0(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{n+n\alpha}(-q;q)_{2n+1}, \tag{4.5}$$

and

$$\bar{\phi}_1(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{n+n\alpha}(-q;q)_{2n}. \tag{4.6}$$

For $t = 0$ and $\alpha = 1$, the generalized functions defined in (4.1)-(4.4) reduce to mock theta functions $\bar{\psi}_0(q)$, $\bar{\psi}_1(q)$ $\bar{\psi}_2(q)$ and $\bar{\psi}_3(q)$, respectively. For $t = 0$, $\alpha = 0$ the generalized functions defined in (4.5)-(4.6) reduce to the mock theta functions $\bar{\phi}_0(q)$ and $\bar{\phi}_1(q)$, respectively.

Theorem 1

$\bar{\psi}_0(t,\alpha,q)$, $\bar{\psi}_1(t,\alpha,q)$, $\bar{\psi}_2(t,\alpha,q)$, $\bar{\psi}_3(t,\alpha,q)$, $\bar{\phi}_0(t,\alpha,q)$ and $\bar{\phi}_1(t,\alpha,q)$ are F_q-functions.

We shall give the proof for $\bar{\psi}_0(t,\alpha,q)$ only. The proofs for the other functions are similar, hence omitted.

Proof

Applying the difference operator $D_{q,t}$, we have

$$t\,D_{q,t}\ \bar{\psi}_0(t,\alpha,q) = \bar{\psi}_0(t,\alpha,q) - \bar{\psi}_0(tq,\alpha,q)$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}} - \frac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(tq)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}}$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}} - \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha}(1-tq^n)}{(-q;q)_{2n}}$$
$$= \frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n(\alpha+1)}}{(-q;q)_{2n}}.$$

So

$$D_{q,t}\ \bar{\psi}_0(t,\alpha,q) = \bar{\psi}_0(t,\alpha+1,q)$$

Hence $\bar{\psi}_0(t,\alpha,q)$ is a F_q-function. By similar working all the generalized functions given in Theorem 1 are F_q-functions.

5. Relation between the Generalized F_q -functions $\bar{\psi}_0(t,\alpha,q), \bar{\psi}_1(t,\alpha,q)$ and $\bar{\phi}_0(t,\alpha,q), \bar{\phi}_1(t,\alpha,q)$

Theorem 2

$$D^2_{q,t}\bar{\psi}_0(t,\alpha,q) = \bar{\psi}_1(t,\alpha,q) + qD^2_{q,t}\ \bar{\psi}_1(t,\alpha,q). \quad (5.1)$$

Proof

Now

$$D^2_{q,t}\bar{\psi}_0(t,\alpha,q) = \bar{\psi}_0(t,\alpha+2,q)$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n(\alpha+2)}}{(-q;q)_{2n}}$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+n+n\alpha}}{(-q;q)_{2n+1}}(1+q^{2n+1})$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+n+n\alpha}}{(-q;q)_{2n+1}} + \frac{q}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+3n+n\alpha}}{(-q;q)_{2n+1}}$$
$$= \bar{\psi}_1(t,\alpha,q) + qD^2_{q,t}\ \bar{\psi}_1(t,\alpha,q),$$

which proves Theorem 2.

Theorem 3

$$\bar{\phi}_0(t,\alpha,q) = \bar{\phi}_1(t,\alpha,q) + q\ D^2_{q,t}\ \bar{\phi}_1(t,\alpha,q). \quad (5.2)$$

Proof

By definition

$$\bar{\phi}_0(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{n+n\alpha}(-q;q)_{2n+1}$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{n+n\alpha}(1+q^{2n+1})(-q;q)_{2n}$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{n+n\alpha}(-q;q)_{2n} + \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}(t)_n q^{3n+n\alpha+1}(-q;q)_{2n}$$
$$= \bar{\phi}_1(t,\alpha,q) + q\ D^2_{q,t}\ \bar{\phi}_1(t,\alpha,q),$$

which proves Theorem 3.

6. Another Generalization of $\bar{\psi}_0(q)$, $\bar{\psi}_1(q)$ and are F_q -functions

We now give another generalization for $\bar{\psi}_0(q)$, $\bar{\psi}_1(q)$ and define :

$$\bar{\psi}_0(t,z,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2}z^{4n}}{(-q;q)_{2n}}, \quad (6.1)$$

and

$$\bar{\psi}_1(t,z,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n}z^{4n+2}}{(-q;q)_{2n+1}}. \quad (6.2)$$

If we put $t = 0$, $z=1$ in (6.1) and (6.2), the generalized functions reduce to mock theta functions $\bar{\psi}_0(q)$ and $\bar{\psi}_1(q)$ respectively.

By taking $z = q^{\alpha/8}$ in (6.1) and (6.2), it can be shown, as is done in section 4, that they are F_q-functions.

7. Relations between Generalized Functions

Theorem 4

$$\bar{\psi}_0(t,\sqrt{z},q) + \bar{\psi}_0(t,\sqrt{zq},q) = \frac{2}{(t)_\infty} + zq\bar{\psi}_1(tq,\sqrt{zq},q) \quad (7.1)$$

and

$$\bar{\psi}_1(t,\sqrt{z},q) + \bar{\psi}_1(t,\sqrt{zq},q) = z\ \bar{\psi}_0(t,\sqrt{zq},q) \quad (7.2)$$

Proof

Writing for $z^{\frac{1}{2}}$ for z in (6.1) and (6.2), we have

$$\bar{\psi}_0(t,\sqrt{z},q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2}z^{2n}}{(-q;q)_{2n}}$$

and

$$\bar{\psi}_1(t,\sqrt{z},q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n}z^{2n+1}}{(-q;q)_{2n+1}}.$$

So

$$\bar{\psi}_0(t,\sqrt{z},q) + \bar{\psi}_0(t,\sqrt{zq},q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2}z^{2n}}{(-q;q)_{2n}} + \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n}z^{2n}}{(-q;q)_{2n}}$$
$$= \frac{1}{(t)_\infty}\left[2+\sum_{n=1}^{\infty}\left\{\frac{(t)_n q^{2n^2}z^{2n}}{(-q;q)_{2n}} + \frac{(t)_n q^{2n^2+2n}z^{2n}}{(-q;q)_{2n}}\right\}\right]$$
$$= \frac{1}{(t)_\infty}\left[2+\sum_{n=1}^{\infty}\frac{(t)_n q^{2n^2}z^{2n}}{(-q;q)_{2n-1}}\right]$$
$$= \frac{1}{(t)_\infty}\left[2+\sum_{n=0}^{\infty}\frac{(t)_{n+1} q^{2n^2+4n+2}z^{2n+2}}{(-q;q)_{2n+1}}\right]$$
$$= \frac{2}{(t)_\infty} + \frac{zq}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(tq)_n q^{2n^2+2n}(zq)^{2n+1}}{(-q;q)_{2n+1}}$$
$$= \frac{2}{(t)_\infty} + zq\bar{\psi}_1(tq,\sqrt{zq},q),$$

which is (7.1).

Again

$$\bar{\psi}_1(t,\sqrt{z},q) + \bar{\psi}_1(t,\sqrt{zq},q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n}z^{2n+1}}{(-q;q)_{2n+1}} + \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+4n+1}z^{2n+1}}{(-q;q)_{2n+1}}$$
$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n}(1+q^{2n+1})z^{2n+1}}{(-q;q)_{2n+1}}$$
$$= \frac{z}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2}(zq)^{2n}}{(-q;q)_{2n}}$$
$$= z\ \bar{\psi}_0(t,\sqrt{zq},q),$$

which is (7.2).

Theorem 5

$$\overline{\psi}_0(t,iz,q) = \overline{\psi}_0(t,z,q) \tag{7.3}$$

and

$$\overline{\psi}_1(t,iz,q) = -\overline{\psi}_1(t,z,q) \tag{7.4}$$

Proof

$$\overline{\psi}_0(t,iz,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2} z^{4n}}{(-q;q)_{2n}}$$

$$= \overline{\psi}_0(t,z,q),$$

which is (7.3).

Again

$$\overline{\psi}_1(t,iz,q) = -\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2+2n} z^{4n+2}}{(-q;q)_{2n+1}}$$

$$= -\overline{\psi}_1(t,z,q),$$

which is (7.4).

8. *q*-Integral Representation for the Generalized F_q-functions

The q-integral was defined by Thomae and Jackson [11, p. 19] as

$$\int_0^1 f(t)\, d_q t = (1-q)\sum_{n=0}^{\infty} f(q^n) q^n .$$

Theorem 6

(i) $\overline{\psi}_0(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\psi}_0(0,aw,q)\, d_q w$,

(ii) $\overline{\psi}_1(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\psi}_1(0,aw,q)\, d_q w$,

(iii) $\overline{\psi}_2(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\psi}_2(0,aw,q)\, d_q w$,

(iv) $\overline{\psi}_3(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\psi}_3(0,aw,q)\, d_q w$,

(v) $\overline{\phi}_0(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\phi}_0(0,aw,q)\, d_q w$,

(vi) $\overline{\phi}_1(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\phi}_1(0,aw,q)\, d_q w$.

Proof

We give a detailed proof of Theorem 6(i) only. The proofs for Theorem 6 (ii)-6(vi) are on the same line, hence omitted.

Limiting case of q-beta integral [11, p.19 (1.11.7)] is

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^\infty t^{x-1}(tq;q)_\infty\, d_q t . \tag{8.1}$$

Now

$$\overline{\psi}_0(t,\alpha,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}}$$

Replacing t by q^t and q^α by a, we have

$$\overline{\psi}_0(q^t,\alpha,q) = \frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n q^{2n^2-n+n\alpha}}{(-q;q)_{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{q^{2n^2-n+n\alpha}}{(-q;q)_{2n}(q^{n+t})_\infty}$$

$$= \sum_{n=0}^{\infty}\frac{q^{2n^2-n+n\alpha}}{(-q;q)_{2n}}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{n+t-1}(wq;q)_\infty\, d_q w \quad \text{by (8.1)}$$

$$= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty \sum_{n=0}^{\infty}\frac{q^{2n^2-n}(aw)^n}{(-q;q)_{2n}}\, d_q w . \tag{8.2}$$

But

$$\overline{\psi}_0(0,\alpha,q) = \sum_{n=0}^{\infty}\frac{q^{2n^2-n+n\alpha}}{(-q;q)_{2n}},$$

and since $q^\alpha = a$,

$$\overline{\psi}_0(0,\alpha,q) = \sum_{n=0}^{\infty}\frac{q^{2n^2-n}(a)^n}{(-q;q)_{2n}} .$$

Hence

$$\overline{\psi}_0(0,aw,q) = \sum_{n=0}^{\infty}\frac{q^{2n^2-n}(aw)^n}{(-q;q)_{2n}} . \tag{8.3}$$

By (8.3), (8.2) can be written as

$$\overline{\psi}_0(q^t,\alpha,q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\ \overline{\psi}_0(0,aw,q)\, d_q w ,$$

which proves (i).

9. Multibasic *q*-Hypergeometric Series Expansions for Generalized Functions

We shall be using the following summation formula [11, (3.6.7), p. 71] and [17, Lemma 10, p. 57] in writing the multibasic expansions of the generalized functions:

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_k q^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k}$$

$$= \sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_m}{(ap/c,bcp;p)_m(q,aq/b;q)_m}\alpha_m . \tag{9.1}$$

Corollary 1

Letting $q \to q^5$ and $c \to \infty$ in (9.1), we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{5k})(1-bp^kq^{-5k})(a,b;p)_k q^{\frac{5k^2+5k}{2}}}{(1-a)(1-b)(q^5,aq^5/b;q^5)_k b^k p^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$

$$= \sum_{m=0}^{\infty}\frac{(ap,bp;p)_m q^{\frac{5m^2+5m}{2}}}{(q^5,aq^5/b;q^5)_m b^m p^{\frac{m^2+m}{2}}}\alpha_m . \tag{9.2}$$

Corollary 2

Letting $q\to q^4$ and $c\to\infty$ in (9.1), we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{4k})(1-bp^kq^{-4k})(a,b;p)_kq^{2k^2+2k}}{(1-a)(1-b)(q^4,aq^4/b;q^4)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{2m^2+2m}}{(q^4,aq^4/b;q^4)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m\,. \quad (9.3)$$

Corollary 3

Letting $q\to q^2$ and $c\to\infty$ in (9.1), we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{2k})(1-bp^kq^{-2k})(a,b;p)_kq^{k^2+k}}{(1-a)(1-b)(q^2,aq^2/b;q^2)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{m^2+m}}{(q^2,aq^2/b;q^2)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m\,. \quad (9.4)$$

Theorem 7

The generalized functions have the following multibasic hypergeometric series expansion:

(i) $$\bar{\psi}_0(t,z,q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+2})(t;q)_{k-1}q^{2k^2}z^{4k}}{(1-q^{k+2})(-q;q^2)_k(-q^2;q^2)_k}$$
$$\times\ \phi\begin{bmatrix}q,0:0,0:tq^{5k+2},q^{5k+5}: \\ q^{k+3}:-q^{2k+1},-q^{2k+2}:0,0:\end{bmatrix};q,q^2,q^5;z^4\end{bmatrix}$$

(ii) $$\bar{\psi}_1(t,z,q)=\frac{z^{-2}}{(1+q)}\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+1})(t;q)_{k-1}q^{2k^2+2k}z^{4k}}{(1-q^{k+1})(-q^2;q^2)_k(-q^3;q^2)_k}$$
$$\times\ \phi\begin{bmatrix}q,0:0,0:tq^{5k+3},q^{5k+5}: \\ q^{k+2}:-q^{2k+2},-q^{2k+3}:0,0:\end{bmatrix};q,q^2,q^5;qz^4\end{bmatrix}.$$

(iii) $$\bar{\psi}_0(t,\alpha,q)=-\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1+q^{2k}+q^{4k})(t;q)_kq^{2k^2-3k+k\alpha}}{(-q;q)_{2k}}$$
$$\times\ \phi\begin{bmatrix}q,tq^k:-q^{2k+2}: \\ 0,0,0:-q^{2k+1}:\end{bmatrix};q,q^2;q^{2k+\alpha-1}\end{bmatrix}.$$

(iv) $$\bar{\psi}_1(t,\alpha,q)=-\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1+q^{2k}+q^{4k})(t;q)_kq^{2k^2-k+k\alpha}}{(-q;q)_{2k+1}}$$
$$\times\ \phi\begin{bmatrix}q,tq^k:-q^{2k+2}: \\ 0,0,0:-q^{2k+3}:\end{bmatrix};q,q^2;q^{2k+\alpha+1}\end{bmatrix}.$$

(v) $$\bar{\psi}_2(t,\alpha,q)=-\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-q^{6k-1})(t;q)_k(q;q^2)_kq^{2k^2-k+k\alpha+1}}{(1-q^{2k+1})(q^4;q^4)_k(-q;q^2)_k}$$
$$\times\ \phi\begin{bmatrix}q,tq^k:q^{2k+1}: \\ 0,0,0:q^{2k+3}:\end{bmatrix};q,q^2;q^{2k+\alpha+2}\end{bmatrix}.$$

(vi) $$\bar{\psi}_3(t,\alpha,q)=-\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1+q^{3k})(t;q)_k(-q;q)_{k-1}q^{k^2-2k+k\alpha}}{(q;q)_k(q;q^2)_k}$$
$$\times\ \phi\begin{bmatrix}q,tq^k:q^{2k+2}: \\ 0,0:q^{2k+1}:\end{bmatrix};q,q^2;q^{k+\alpha-1}\end{bmatrix}.$$

Proof of (i)

We shall give the proof for $\bar{\psi}_0(t,z,q)$ in detail, for other functions we shall only give the value of the parameters .

Taking $a=\frac{t}{q}$, $b=q^2$, $p=q$ and $\alpha_m=\frac{(q^5;q^5)_m(tq^2;q^5)_mz^{4m}}{(q^3;q)_m(-q;q^2)_m(-q^2;q^2)_m}$ in (9.2), we have

$$\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+2})(t/q,q^2;q)_kq^{2k^2}}{(1-t/q)(1-q^2)(tq^2,q^5;q^5)_k}$$
$$\times\sum_{m=0}^{\infty}\frac{(q^5;q^5)_{m+k}(tq^2;q^5)_{m+k}z^{4(m+k)}}{(q^3;q)_{m+k}(-q;q^2)_{m+k}(-q^2;q^2)_{m+k}}=\sum_{m=0}^{\infty}\frac{q^{2m^2}(t;q)_mz^{4m}}{(-q;q^2)_m(-q^2;q^2)_m}\,. \quad (9.5)$$

The right hand side of (9.5) is equal to

$$(t;q)_\infty\ \bar{\psi}_0(t,z,q).$$

The left hand side of (9.5) is equal to

$$\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+2})(t;q)_{k-1}q^{2k^2}z^{4k}}{(1-q^{k+2})(-q;q^2)_k(-q^2;q^2)_k}$$
$$\times\sum_{m=0}^{\infty}\frac{(q^{5k+5};q^5)_m(tq^{5k+2};q^5)_mz^{4m}}{(q^{k+3};q)_m(-q^{2k+1};q^2)_m(-q^{2k+2};q^2)_m}$$
$$=\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+2})(t;q)_{k-1}q^{2k^2}z^{4k}}{(1-q^{k+2})(-q;q^2)_k(-q^2;q^2)_k}$$
$$\times\ \phi\begin{bmatrix}q,0:0,0:tq^{5k+2},q^{5k+5}: \\ q^{k+3}:-q^{2k+1},-q^{2k+2}:0,0:\end{bmatrix};q,q^2,q^5;z^4\end{bmatrix},$$

which proves (i).

Proof of (ii)

Take $a=\frac{t}{q}$, $b=q$, $p=q$ and $\alpha_m=\frac{(q^5;q^5)_m(tq^3;q^5)_mq^mz^{4m}}{(q^2;q)_m(-q^2;q^2)_m(-q^3;q^2)_m}$ in (9.2).

Proof of (iii)

Letting $q\to q^4$, $p\to q^2$, $a=b=1$ and $\alpha_m=\frac{(t;q)_m(-q^2;q^2)_mq^{m^2-2m+m\alpha}}{(-q;q^2)_m}$ in (9.3).

Proof of (iv)

Letting $q\to q^4$, $p\to q^2$, $a=b=1$ and $\alpha_m=\frac{(t;q)_m(-q^2;q^2)_mq^{m^2+m\alpha}}{(-q^3;q^2)_m}$ in (9.3).

Proof of (v)

Letting $q\to q^4$, $p\to q^2$, $a=\frac{1}{q}$, $b=q$ and $\alpha_m=\frac{(t;q)_m(q;q^2)_mq^{m^2+m+m\alpha}}{(q^3;q^2)_m}$ in (9.3).

Proof of (vi)

Letting $q\to q^2$, $p\to q$, $a=b=-1$, and $\alpha_m=\frac{(-1)^m(t;q)_m(q^2;q^2)_mq^{\frac{m^2-3m+2m\alpha}{2}}}{(q;q^2)_m}$ in (9.4).

10. Conclusions

I have given two generalizations of these mock theta functions and shown they are F_q -functions, so they satisfy the properties of the general class of F_q -functions. The generalization helps in giving relationship between these functions. Apart from these values, we can give other values to have another set of functions having these properties.

I think these relations may yield interesting results in the theory of partitions.

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