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Integrability and Exact Solutions of Complexly Coupled Kdv System: PainlevÉ Property and Bäcklund Transformation

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Abstract In this paper, we have analyzed the integrability properties of complexly coupled KdV system, and discuss the integrability properties through Painlevé (P) analysis. Further, we using Bäcklund transformation to obtain exact solutions of complexly coupled KdV system

Keywords Integrability, PainlevÉ Property (PP), Bäcklund Transformation and Exact Solution

1. Introduction

The study of the importance of the connection of integrability and Painlevé property (PP), in the context of ODEs, has drawn much attention in the recent literature[1-7]. In fact, the study has been well utilized, both in the original sense of Ablowitz et al.[3] and in the generalized form of Weiss et al.[5], to describe the connection of PDEs of the Painlevé type and Lax pairs, Bäcklund transformation, symmetry vector fields and recursion operators[8]. However, the technique of singular manifold expansion developed for obtaining the connection of PDEs possessing PP and Lax pairs, Bäcklund transformations, etc. in the case of integrable systems could not receive any attention for its carry over to non-integrable evolution equations. Consequently, we have, herein find the integrability properties associated with the (1 + 1) complexly coupled KdV system[9]. The system can be written in the form:

$$\begin{aligned} &u_t - 6uu_x - 6vv_x - u_{xxx} = 0, \\ &v_t - 6uv_x - 6vu_x - v_{xxx} = 0. \end{aligned} \tag{1.1}$$

2. The Painlevé Analysis

In this part, we study the Painlevé integrability of Eq. (1.1) following the Weiss's algorithm [8] of singularity analysis. To proceed with the Painlevé singularity analysis, set

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j+\alpha}, \quad v = \sum_{j=0}^{\infty} v_j \varphi^{j+\beta} \tag{2.1}$$

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Where ϕ (x,t)=0, the "expansion coefficients" u_j and v_j are analytic functions of independent variables; ϕ (x,t)=0, describe the singular manifold, α, β are an integer to be determined. Insert expansion (2.1) in Eq. (1.1) a leading order analysis uniquely values α and β. Consequently, the said analysis yields:

$$\begin{aligned} &\alpha = \beta = 2, \quad a_0 = -\varphi_x^2, \quad b_0 = \pm\varphi_x^2, \\ &j = -1, 2, 3, 4, 4, 6, \end{aligned} \tag{2.2}$$

The recursion relation for u_j (x,t) and v_j (x,t) are found to be:

$$\begin{aligned} &(j-4)u_{j-2}\varphi_t + u_{j-3,t} - (j-4)(j-3)(j-2)u_j\varphi_x^3 \\ &-3(j-4)(j-3)u_{j-1,x}\varphi_x^2 - 3(j-4)(j-3)u_{j-1}\varphi_x\varphi_{xx} \\ &-3(j-4)u_{j-2,x}\varphi_{xx} - 3(j-4)u_{j-2,xx}\varphi_x \\ &-u_{j-3,xxx} - (j-4)u_{j-2}\varphi_{xxx} - \\ &-6\sum_{m=0}^{j}[u_{j-m}(u_m(m-2)\varphi_x - u_{m-1,x})] \\ &6\sum_{m=0}^{j}[v_{j-m}(v_m(m-2)\varphi_x - v_{m-1,x})] = 0. \end{aligned} \tag{2.3}$$

$$\begin{aligned} &(j-4)v_{j-2}\varphi_t + v_{j-3,t} - (j-4)(j-3)(j-2)v_j\varphi_x^3 \\ &-(j-4)(j-3)v_{j-1,x}\varphi_x^2 - 3(j-4)(j-3)v_{j-1}\varphi_x\varphi_{xx} \\ &-(j-4)v_{j-2,x}\varphi_{xx} - 3(j-4)v_{j-2,xx}\varphi_x \\ &-v_{j-3,xxx} - (j-4)v_{j-2}\varphi_{xxx} - \\ &6\sum_{m=0}^{j}[u_{j-m}(v_m(m-2)\varphi_x - v_{m-1,x})] \\ &-6\sum_{m=0}^{j}[v_{j-m}(u_m(m-2)\varphi_x - u_{m-1,x})] = 0. \end{aligned} \tag{2.4}$$

Assigning j = 0, 1, 2 in Eqs. (2.3) and (2.4), we get
j=0

$$u_0 = -\varphi_x^2, \quad v_0 = \pm\varphi_x^2. \tag{2.5}$$

j=1

$$-\varphi_x^3(2u_1+3v_1-13\varphi_{xx})=0, \quad -\varphi_x^3(3u_1+2v_1-13\varphi_{xx})=0, \tag{2.6}$$

j=2

$$-2u_0\varphi_t-6u_{1x}\varphi_x^2-6(-2u_2u_0\varphi_x+u_1(-u_1\varphi_x-u_{0x})-u_0u_{0x})-6(-2v_2v_0\varphi_x+v_1(-v_1\varphi_x-v_{0x})-v_0v_{0x})-6(u_1\varphi_x\varphi_{xx}+6u_{0x}\varphi_{xx}+\varphi_xu_{0xx}+2u_0\varphi_{xxx}=0. \tag{2.7}$$

$$-2v_0\varphi_t-(v_2(-2v_0\varphi_x^2+v_1(-u_1\varphi_x+u_0)-v_0u_{1x}-6v_{1x}\varphi_x^2-6(-2u_2v_0\varphi_x+u_1(-v_1\varphi_x-v_{0x})-u_0v_{1x})-(u_1\varphi_x\varphi_{xx}+6v_0\varphi_{xx}+\varphi_xv_{0xx}+2v_0\varphi_{xxx}=0. \tag{2.8}$$

j=3

$$-u_1\varphi_t+u_{0t}-6(-2u_3u_0\varphi_x+u_2(-u_1\varphi_x-u_{0x})-u_1u_{1x}+u_0(u_3\varphi_x-u_{2x}))-6(-2v_3v_0\varphi_x+v_2(-v_1\varphi_x-v_{0x})-v_1v_{1x}+v_0(v_3\varphi_x-v_2))+3u_{1x}\varphi_{xx}+u_{1xx}\varphi_x+u_1\varphi_{xxx}-u_{0xxx}=0 \tag{2.9}$$

$$-v_1\varphi_t+v_{0t}-6(-2v_3v_0\varphi_x+v_2(-v_1\varphi_x-v_{0x})-v_1v_{1x}+v_0(v_3\varphi_x-v_{2x}))-6(-2u_3u_0\varphi_x+u_2(-u_1\varphi_x-u_{0x})-u_1u_{1x}+u_0(u_3\varphi_x-u_2))+3v_{1x}\varphi_{xx}+v_{1xx}\varphi_x+v_1\varphi_{xxx}-v_{0xxx}=0 \tag{2.10}$$

j=4

$$u_{1t}-(-2u_0\varphi_x+u_3(-u_1\varphi_x-u_{0x})-u_2u_{1x}+u_1(u_3\varphi_x-u_{2x})+u_0(2u_4\varphi_x-u_3)-(-2v_4v_0\varphi_x+v_3(-v_0\varphi_x-v_x)-v_2v_{1x}+v_1(v_3\varphi_x-v_{2x})+v_0(2v_4\varphi_x-v_{3x}))-u_{1xxx}=0, \tag{2.11}$$

$$v_{1t}-(-2v_3u_0\varphi_x+v_3(-u_1\varphi_x-u_{0x})-v_2u_{1x}+v_1(u_3\varphi_x-u_{2x})+v_0(2u_4\varphi_x-u_3)-(-2u_4v_0\varphi_x+u_3(-v_0\varphi_x-v_x)-u_2v_{1x}+u_1(v_3\varphi_x-v_{2x})+u_0(2v_4\varphi_x-v_{3x}))-v_{1xxx}=0, \tag{2.12}$$

j=5

$$u_3\varphi_t+u_{2t}-6u_5\varphi_x^2-66(-2u_0u_5\varphi_x+u_4(-u_1\varphi_x-u_{0x})-u_3u_{1x}+u_2(u_3\varphi_x-u_{2x})+u_1(2u_4\varphi_x-u_{3x})+u_0(3u_5\varphi_x-u_{4x}))-6\varphi_x^2u_{4x}-6(-2v_0v_5\varphi_x+v_4(-v_1\varphi_x-v_{0x})-v_3v_{1x}+v_2(v_2\varphi_x-v_{2x})+v_1(3v_4\varphi_x-v_{3x})+v_0(3v_5\varphi_x-v_{4x}))-6u_4\varphi_x\varphi_{xx}-3u_3\varphi_{xx}-3u_{3xx}\varphi_x-u_3\varphi_{xxx}-u_{2xxx}=0 \tag{2.13}$$

$$v_3\varphi_t+v_{2t}-6v_5\varphi_x^3-6(-2u_0v_5\varphi_x+v_4(-u_1\varphi_x-u_{0x})-v_3u_{1x}+v_2(u_3\varphi_x-u_{2x})+v_1(2u_4\varphi_x-u_{3x})+v_0(3u_5\varphi_x-u_{4x}))-6\varphi_x^2u_{4x}-6(-2v_0u_5\varphi_x+u_4(-v_1\varphi_x-v_{0x})-u_3v_{1x}+u_2(v_2\varphi_x-v_{2x})+u_1(3v_4\varphi_x-v_{3x})+u_0(3v_5\varphi_x-v_{4x}))-6v_4\varphi_x\varphi_{xx}-3v_3\varphi_{xx}-3v_{3xx}\varphi_x-v_3\varphi_{xxx}-v_{2xxx}=0 \tag{2.14}$$

j=6

$$2u_4\varphi_t+u_{3t}-24u_6\varphi_x^3-6(-2u_6u_0\varphi_x+u_5(-u_1\varphi_x-u_{0x})-u_4u_{1x}+u_3\varphi_x-u_{2x})+u_2(3u_4\varphi_x-u_{3x})+u_1(3u_5\varphi_x-u_{4x})+u_0(4u_6\varphi_x-u_{5x}))-18u_{5x}\varphi_x-6(-2v_6v_0\varphi_x+v_5(-v_1\varphi_x-v_{0x})-v_4v_{1x}+v_3(v_3\varphi_x-v_{2x})+v_2(2v_4\varphi_x-v_{3x})+v_1(3v_5\varphi_x-v_{4x})+v_0(4v_6\varphi_x-v_{5x}))-18u_5\varphi_x\varphi_{xx}-6u_{4x}\varphi_{xx}-6u_4\varphi_{xxx}-u_{3xxx}=0, \tag{2.15}$$

$$v_{3t}-12u_6v_0\varphi_x+6u_5(-2v_1\varphi_x+v_{0x})+6(v_3u_{2x}+v_2u_{3x}+v_1u_{4x}+v_0u_{5x}+u_4(-2v_2\varphi_x+v_{1x})+u_3(-2v_3\varphi_x+v_{2x}))-2(3(4v_6\varphi_x^3-v_5u_{0x}-u_2v_{3x}+u_1(2v_5\varphi_x-v_{4x})+u_0(2v_6\varphi_x-v_{5x})+3v_{5x}\varphi_x^2+3v_5\varphi_x\varphi_{xx}+v_{4x}\varphi_{xx}+v_{4xx}\varphi_x)+v_4(-\varphi_t+6u_2\varphi_x-3u_{1x}+\varphi_{xxx}))-v_{3xxx}=0. \tag{2.16}$$

Therefore, we know for resonance Laurent series (2.1) admits the sufficient number of arbitrary functions. So it is concluded that the complexly coupled KdV system (1.1) possesses Painlevé property.

3. Bäcklund Transformation and Special Solutions

Up to know, we have determined that the coupled KdV system is integrable in the sense of having Painlevé property. To obtain the more integrable information of this equation, in the following, we give the Bäcklund transformation and some special solutions of this system. If we now apply the usual technique of determination of the "resonances" by cutting of series (2.1), we set u_j=v_j=0 and i≥3, then we have:

$$u=\frac{u_0}{\varphi^2}+\frac{u_1}{\varphi}+u_2, \quad v=\frac{v_0}{\varphi^2}+\frac{v_1}{\varphi}+v_2 \tag{2.17}$$

By solving Eqs. (2.6) we get

$$u_1=\frac{13}{5}\varphi_{xx}, \quad v_1=\frac{13}{5}\varphi_{xx}. \tag{2.18}$$

So we obtain the Bäcklund transformation of Eq. (1.1)

$$u=\frac{-\varphi_x^2}{\varphi^2}+\frac{13}{2}\frac{\varphi_{xx}}{\varphi},$$
$$v=\frac{-\varphi_x^2}{\varphi^2}+\frac{13}{2}\frac{\varphi_{xx}}{\varphi}. \tag{2.19}$$

In the following, we derive the explicit special solutions for (1.1) by the Bäcklund transformation (2.19). To do this, we take two cases of φ:

Case I: φ =Exp(wx+ct)+Exp(-(wx+ct)) we obtain the soliton solution of Eq. (1.1)

$$u=\frac{w^2}{5}(8+5\ \mathrm{sech}(wx+ct)),$$
$$v=\frac{w^2}{5}(8+5\ \mathrm{sech}(wx+ct)), \tag{2.20}$$
$$where\ \ c=\frac{116w^3}{5}$$

Case II: ϕ =Exp(wx+ct)-Exp(-(wx+ct)) we can write the solution of Eq. (1.1) in the form

$$u=\frac{w^2}{5}(8-5\ \mathrm{csch}(wx+ct)),$$
$$v=\frac{w^2}{5}(8-5\ \mathrm{csch}(wx+ct)), \tag{2.21}$$
$$where\ \ c=\frac{116w^3}{5}$$

4. Conclusions

Thus, the Painlevé analysis is one of the most important methods to identify the integrability cases and the integrability properties of complexly coupled KdV system. In this paper, we applied the Painlevé analysis to complexly coupled KdV system and established their integrability properties. Also, by truncating the Laurent series at the constant level term, we have constructed the Bäcklund transformation. We obtained exact solutions of the system.

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