

# On the Clenshaw Method for Solving Linear Ordinary Differential Equations

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**Abstract** In this paper we introduce, briefly, Clenshaw method which is a kind of spectral method and then by exploiting the trigonometric identity property of Chebyshev polynomial in this method we try to get more accurate approximate solution of linear differential equations. We compare the results by some numerical examples.

**Keywords** Clenshaw Method, Pseudo-Clenshaw Method, Chebyshev Polynomials

## 1. Introduction

Spectral methods arise from the fundamental problem of approximation of a function by interpolation on an interval, and are very much successful for the numerical solution of ordinary or partial differential equations[1]. Since the time of Fourier (1882), spectral representations in the analytic study of differential equations have been used and their applications for numerical solution of ordinary differential equations refer, at least, to the time of Lanczos[2].

Spectral methods have become increasingly popular, especially, since the development of Fast transform methods, with applications in problems where high accuracy is desired. A survey of some applications is given in[3].

The basis of spectral methods to solve differential equations is to expand the solution function as a finite series of very smooth basis functions, as follows

$$y_N(x) = \sum_{n=0}^N a_n \phi_n(x) \quad (1)$$

in which,  $\phi_n$  is one of choice of the eigenfunctions of a singular Sturm-Liouville problem. If the solution is infinitely smooth, the convergence of spectral method is more rapid than any finite power of  $1/N$ . That is the produced error of approximation (1), when  $N \rightarrow \infty$ , approaches zero with exponential rate[1]. This phenomenon is usually referred to as "spectral accuracy"[3]. The accuracy of derivatives obtained by direct, term by term differentiation of such truncated expansion naturally deteriorates[1]. Although there will be problem but for high order derivatives truncation and round off errors may deteriorate, but for low order derivatives and sufficiently high-order truncations this deterioration is negligible. So, if the solution function and coefficient

functions of the differential equation are analytic on  $[a, b]$ , spectral methods will be very efficient and suitable. We call function  $y$  is analytic on  $[a, b]$  if is infinitely differentiable and with all its derivatives on this interval are bounded variation.

In next section, first, we introduce Clenshaw method and then by exploiting the trigonometric identity property of Chebyshev polynomial, we develop a numerical scheme referred to as Pseudo-Clenshaw method.

## 2. Procedures

### (i)-Clenshaw method

Consider the following differential equation:

$$Ly = \sum_0^M f_{M-i}(x) D^i y = f(x), \quad x \in [-1, 1], \quad (2)$$

$$By = C \quad (3)$$

where  $L = \sum_0^M f_{M-i}(x) D^i$ , and  $f_i$ ,  $i = 0, 1, \dots, M$ ,  $f$ , are

known real functions of  $x$ ,  $D^i$  denotes  $i^{\text{th}}$  order of differentiation with respect to  $x$ ,  $B$  is a linear functional of rank  $M$  and  $C \in \mathfrak{R}^M$ .

Here (3) can be initial, boundary or mixed conditions. The basis of spectral methods to solve this class of equations is to expand the solution function,  $y$ , in (2) and (3) as a finite series of very smooth basis functions, as given below

$$y_N(x) = \sum_{n=0}^N a_n T_n(x) \quad (4)$$

where,  $\{T_n(x)\}_0^N$  is sequence of Chebyshev polynomials of the first kind. By replacing  $y_N$  in (2), we define the residual term by  $r_N(x)$  as follows

$$r_N(x) = Ly_N - f \quad (5)$$

In spectral methods, the main target is to minimize  $r_N(x)$

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Published online at <http://journal.sapub.org/ajcam>

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throughout the domain as much as possible with regard to (3), and in the sense of point-wise convergence. Implementation of these methods leads to a system of linear equations with  $N + 1$  equations and  $N + 1$  unknowns  $a_0, a_1, \dots, a_N$ .

Consider the following differential equation:

$$P(x)y'' + Q(x)y' + R(x)y = S(x), x \in (-1, 1), \tag{6}$$

$$y(-1) = \alpha, y(1) = \beta.$$

First, for an arbitrary natural number  $N$ , we suppose that the approximate solution of equations (6) is given by (4). Our target is to find  $\underline{a} = (a_0, a_1, \dots, a_N)^t$ . For this reason, we put

$$P(x) \cong \sum_{i=0}^N \xi_i T_i(x), Q(x) \cong \sum_{i=0}^N \gamma_i T_i(x), R(x) \cong \sum_{i=0}^N \lambda_i T_i(x). \tag{7}$$

Using this fact that the Chebyshev expansion of a function  $u \in L_w^2(-1, 1)$  is

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x); \hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^1 u(x) T_k(x) w(x) dx, \text{ we}$$

can find coefficients  $\xi_i, \gamma_i$  and  $\lambda_i$  as follows:

$$\xi_i = \frac{2}{\pi c_i} \int_{-1}^1 \frac{P(x)T_i(x)}{\sqrt{1-x^2}} dx$$

$$\gamma_i = \frac{2}{\pi c_i} \int_{-1}^1 \frac{Q(x)T_i(x)}{\sqrt{1-x^2}} dx \tag{8}$$

$$\lambda_i = \frac{2}{\pi c_i} \int_{-1}^1 \frac{R(x)T_i(x)}{\sqrt{1-x^2}} dx,$$

where,  $c_0 = 2$  and  $c_i = 1$  for  $i \geq 1$ .

To compute the right-hand side of (8) it is sufficient to use an appropriate numerical integration method. Here, we use  $(N + 1)$  - point Gauss - Chebyshev - Lobatto quadrature

$$x_j = \cos \frac{\pi j}{N}, w_j = \frac{\pi}{\tilde{c}_j N}, 0 \leq j \leq N,$$

where  $\tilde{c}_0 = \tilde{c}_N = 2$  and  $\tilde{c}_j = 1$  for  $j = 1, 2, \dots, N - 1$ .

Note that, for simplicity of the notation, these points are arranged in descending order, namely,

$$x_N < x_{N-1} < \dots < x_1 < x_0,$$

with weights

$$w_k = \frac{\pi}{N}, 1 \leq k \leq N - 1,$$

$$= \frac{\pi}{2N}, k = 0, k = N,$$

and nodes  $x_k = \cos \frac{\pi k}{N}, k = 0, 1, \dots, N$ . That is, we put[4]:

$$\xi_i \cong \frac{\pi}{N} \sum_{k=0}^N P(\cos(\frac{k\pi}{N})) T_i(\cos(\frac{k\pi}{N})),$$

and using  $T_i(x) = \cos(i \cos^{-1} x)$ , we get

$$\xi_i \cong \frac{\pi}{N} \sum_{k=0}^N P(\cos(\frac{k\pi}{N})) \cos(\frac{\pi ik}{N}),$$

where, notation  $\sum$  means first and last terms become

half. Therefore, we will have:

$$\xi_i \cong \frac{\pi}{N} \sum_{k=0}^N P(\cos(\frac{k\pi}{N})) \cos(\frac{\pi ik}{N}),$$

$$\gamma_i \cong \frac{\pi}{N} \sum_{k=0}^N Q(\cos(\frac{k\pi}{N})) \cos(\frac{\pi ik}{N}), \tag{9}$$

$$\lambda_i \cong \frac{\pi}{N} \sum_{k=0}^N R(\cos(\frac{k\pi}{N})) \cos(\frac{\pi ik}{N}).$$

Now, substituting (4) and (9) in equations (6), and using the fact that

$$y'(x) \cong \sum_{m=0}^N a_m^{(1)} T_m(x), a_m^{(1)} = \frac{2}{c_m} \sum_{\substack{p=m+1 \\ m+p=\text{odd}}}^N p a_p, m = 0, 1, \dots, N - 1, a_N^{(1)} = 0,$$

$$y''(x) \cong \sum_{m=0}^N a_m^{(2)} T_m(x), a_m^{(2)} = \frac{1}{c_m} \sum_{p=m+2}^N p(p^2 - m^2) a_p,$$

$$m = 0, 1, \dots, N - 2, a_{N-1}^{(2)} = a_N^{(2)} = 0,$$

$$m+p=\text{even}$$

in this manner, we get

$$\sum_{i=0}^N \sum_{m=0}^N \xi_i a_m^{(2)} T_i(x) T_m(x) + \sum_{i=0}^N \sum_{m=0}^N \gamma_i a_m^{(1)} T_i(x) T_m(x)$$

$$+ \sum_{i=0}^N \sum_{m=0}^N \lambda_i a_m T_i(x) T_m(x) = S(x), \tag{10}$$

$$\sum_{i=0}^N a_i T_i(-1) = \alpha, \sum_{i=0}^N a_i T_i(1) = \beta. \tag{11}$$

Now, we multiply both sides of (10) by  $\frac{2}{\pi c_j} \frac{T_j(x)}{\sqrt{1-x^2}}$ , and

integrate from -1 to 1, to obtain

$$\frac{2}{\pi c_j} \sum_{i=0}^N \sum_{m=0}^N [\xi_i a_m^{(2)} + \gamma_i a_m^{(1)} + \lambda_i a_m] \int_{-1}^1 \frac{T_i(x) T_m(x) T_j(x)}{\sqrt{1-x^2}} dx$$

$$= \frac{2}{\pi c_j} \int_{-1}^1 \frac{S(x) T_j(x)}{\sqrt{1-x^2}} dx, j = 0, 1, \dots, N - 2, \tag{12}$$

where,

$$\int_{-1}^1 \frac{T_i(x) T_m(x) T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & i = m = j = 0, \\ \frac{\pi}{2} \delta_{i,m}, & i + m > 0, j = 0, \\ \frac{\pi}{4} (\delta_{j,i+m} + \delta_{j,|i-m|}), & j > 0, \end{cases} \tag{13}$$

with,  $\delta_{i,j} = 1$ , when  $i = j$ , and zero when  $i \neq j$  [5].

We can also compute the integrals in the right-hand side of (12) by the method of numerical integration using  $N + 1$  -point Gauss-Chebyshev-Lobatto quadrature. Therefore, substituting (13) in (12) and using the fact that  $T_i(\pm 1) = (\pm 1)^i$ , equations (12) and (11) make a system of  $N + 1$  equations for  $N + 1$  unknowns  $a_0, a_1, \dots, a_N$ , and we can obtain  $(a_0, a_1, \dots, a_N)^t$  from this system.

(ii)-Pseudo-Cleenshaw method

We assume that an approximate solution to Eq. (6) is given by

$$\bar{y}(x) = \sum_{n=0}^N a_n T_n(x). \tag{14}$$

Recall the Chebyshev polynomial given by:

$$T_n(x) = \cos(ncos^{-1}(x)).$$

Let  $\theta = \cos^{-1}(x)$ , then  $T_n(\theta) = \cos n\theta$ . By using this identity (14) becomes

$$\bar{y}(x) = \sum_{n=0}^N a_n \cos n\theta. \tag{15}$$

The first and second derivatives of (15) are given, respectively, as

$$\bar{y}'(x) = \sum_{n=0}^N a_n \left( \frac{nsin n\theta}{sin \theta} \right),$$

$$\bar{y}''(x) = \sum_{n=0}^N a_n \left( \frac{nsin n\theta \cos \theta - n^2 \cos n\theta \sin \theta}{sin^3 \theta} \right).$$

Substituting  $\bar{y}(x), \bar{y}'(x)$  and  $\bar{y}''(x)$  in Eq. (6) with the functions  $P, Q, R$  and  $S$  in term of  $\theta$ , we get

$$\bar{P}(\theta) \sum_{n=0}^N a_n \left( \frac{nsin n\theta \cos \theta - n^2 \cos n\theta \sin \theta}{sin^3 \theta} \right) + \bar{Q}(\theta) \sum_{n=0}^N a_n \left( \frac{nsin n\theta}{sin \theta} \right) +$$

$$\bar{R}(\theta) \sum_{n=0}^N a_n \cos n\theta = \bar{S}(\theta),$$

where  $\theta \in [-\pi, \pi]$ ,  $\bar{y}(-\pi) = \alpha, \bar{y}(\pi) = \beta$  and

$$\bar{P}(\theta) \equiv \sum_{i=0}^N \xi_i T_i(\theta), \quad \bar{Q}(\theta) \equiv \sum_{i=0}^N \gamma_i T_i(\theta), \quad \bar{R}(\theta) \equiv \sum_{i=0}^N \lambda_i T_i(\theta).$$

Some more substitutions through 8-13 lead to desire relations.

### 3. Numerical Examples

Now, we consider some examples with Clenshaw and Pseudo-Clenshaw methods and observe the power of this method comparing with usual numerical methods such as Euler's or Runge-Kutta's, Adams methods.

*Problem 1.* Consider

$$y''(x) + 3xy'(x) + x^4 y(x) = 6x + 9x^3 + x^7, \quad x \in [-1, 1],$$

$$y(-1) = -1, \quad y(1) = 1,$$

with exact solution  $y(x) = x^3$ .

We solved it by Runge-Kutta with orders two and four and also Adams method. For these methods we used, the same step size and step number. The maximum errors were  $1.5 \times 10^{-5}, 1.8 \times 10^{-7}, 1.2 \times 10^{-5}$ , respectively. We, also, solved it by shooting method with the same step size for steps  $N=14, 17$ . We had maximum errors  $2.9 \times 10^{-5}, 1.3 \times 10^{-5}$ , respectively. As we see the rate of improvement of accuracy is very low. But we used the Clenshaw method with Chebyshev e basis for  $N=14$  and  $N=17$ . The maximum errors were about,  $2.2 \times 10^{-13}$ , and  $3.4 \times 10^{-14}$ , respectively, and we used the Pseudo- Clenshaw method with  $N=14, 17$ , and maximum errors were  $8.9 \times 10^{-16}$ , and  $4.4 \times 10^{-16}$ , respectively. As we can see, Clenshaw and Pseudo- Clenshaw methods for solving such problems have high rate of convergency. Existence of  $x^7$ , indicates when  $N$  get the value 7, the error becomes zero. If we observe above errors they are rounding

errors.

*Example 2.* Let us consider

$$y''(x) + xy'(x) + y = x \cos(x), \quad x \in [-1, 1],$$

$$y(-1) = \sin(-1), \quad y(1) = \sin(1),$$

with the exact solution  $y(x) = \sin(x)$ . This example was chosen from[6]. We solved it by Runge-Kutta with orders two and four and also Adams method. The maximum errors are  $2.5 \times 10^{-4}, 2.4 \times 10^{-7}, 1.1 \times 10^{-5}$ , respectively. That is, these methods give good results for such problems. For these methods we used the same step size and step number. We also solved it by the Clenshaw and Pseudo-Clenshaw methods with  $N=5, 8, 16$ , the maximum errors produced from this method are given in Table 1, where  $y_c(x)$  and  $y_{pc}(x)$  mean the Clenshaw and Pseudo-Clenshaw methods, respectively.

**Table 1.**

$N$	$\ y(x) - \tilde{y}_c(x)\ _\infty$	$\ y(x) - \tilde{y}_{pc}(x)\ _\infty$
5	$2.11 \times 10^{-5}$	$1.10 \times 10^{-5}$
8	$5.71 \times 10^{-8}$	$0.01 \times 10^{-9}$
16	$1.11 \times 10^{-16}$	$8.98 \times 10^{-17}$

*Problem 3:* Consider

$$y'' + xy' - y = f(x), \quad x \in (-1, 1),$$

$$y(\pm 1) = e^{\pm 5} + \sin(1),$$

where,

$$f(x) = (24 + 5x)e^{5x} + (2 + 2x^2) \cos(x^2) - (4x^2 + 1) \sin(x^2),$$

so that the exact solution is  $y(x) = e^{5x} + \sin(x^2)$ .

For comparison, we solved this problem by finite difference method, using the central differences for the derivatives. The mesh points are given by  $x_i = -1 + ih, h = \frac{2}{N}$ . The maximum errors given by this method are,

$$3.100, 7.898 \times 10^{-1}, 1.984 \times 10^{-1}, 4.968 \times 10^{-2},$$

$$1.242 \times 10^{-2}, 3.106 \times 10^{-3}$$

for  $N=16, 32, 64, 128, 256, 512$ , respectively.

We solved it by the Clenshaw and Pseudo-Clenshaw methods with  $N=10, 11, 12$ , the maximum errors produced from this method are given in Table 2 shows the results of solving this problem by these methods.

**Table 2.**

$N$	$\ y(x) - \tilde{y}_c(x)\ _\infty$	$\ y(x) - \tilde{y}_{pc}(x)\ _\infty$
10	$1.07 \times 10^{-2}$	$8.22 \times 10^{-3}$
11	$2.20 \times 10^{-3}$	$9.83 \times 10^{-4}$
12	$6.14 \times 10^{-4}$	$7.24 \times 10^{-5}$

### 4. Conclusions

Results in these examples show the efficiency of Pseudo-Clenshaw method for obtaining a better numerical result. Unfortunately, for equations with non-analytical coefficient

functions these methods have low convergence, because when  $N$  increases the rate of improvement of accuracy is very low. This is because of the lack of smoothness of the coefficient function. But, when we solved it by the pseudo-spectral method, since coefficient functions do not need expansion in the form of (9), the error produced from using this method, will be better than Clenshaw and Pseudo-Clenshaw methods[6].

Author next goal is to work more on this method to get better results.

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