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Similarity Solutions for Reactive Shock Hydrodynamics

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Abstract The method of Lie group invariance is used to obtain a class of self-similar solutions for a one-dimensional, time-dependent problem in shock hydrodynamics, with a chemical reaction taking place behind the shock. The forms of the initial specific volume v_0 and the reaction rate Q, for which the problem is invariant and admits self-similar solutions, are also found.

Keywords Lie Group, Reactive Shock Hydrodynamics, Similarity Solutions, Shock Waves, Invariance, Non-Uniform Medium

1. Introduction

Many flow fields involving wave phenomena are governed by quasi linear hyperbolic system of partial differential equations (PDEs). For nonlinear systems involving discontinuities such as shocks, we do not generally have the complete exact solutions, and we have to rely on some approximate analytical or numerical methods which may be useful to provide information to understand the physics involved. One of the most powerful methods to obtain the similarity solutions to PDEs is similarity method which is based upon the study of their invariance with respect to one parameter Lie group of transformations. Indeed, with the help of infinitesimals and invariant surface conditions, one can construct similarity variables which can reduce these PDEs to ordinary differential equations (ODEs).

The physical situation that motivates this study is a hydrodynamic medium in which a chemical reaction occurs. The reaction is initiated by a plane shock wave which is introduced into the medium at time $t = 0$ by a driving piston. For detonation waves, it is experimentally observed that after a time, a steady-state condition is reached from the viewpoint of an observer riding on the shock. This problem was first studied by Chapman[1] and Jouget[2], who assumed that the chemical reaction takes place instantaneously in the shock front. Later, their theory was refined by Zeldovich[3], J. Von Neumann[4] and Doering[5], to include a zone of finite width behind the shock, where chemical reaction occurs. A thorough discussion can be found in Courant and Friedrichs[6], and Fickett and Davis[7].

Self-similar solutions in non-reactive shock hydrodynamics and gas dynamics have been studied extensively in planar, cylindrical and spherical geometry. We mention the work of Guderley[8], Taylor[9], Sedov[10], Zeldovich and Raizer[11], Sharma and Radha[12], Arora and Sharma [13,14], Sharma and Radha[15] and L. P. Singh et al. [16].

In the present paper, following Bluman and Cole[17], Bluman and Kumei[18], and in a spirit closer to Logan[19,20], we obtain the self-similar solutions to a one-dimensional time- dependent problem in shock hydrodynamics with a chemical reaction taking place. Also, we obtain the form of the initial specific volume v_0 and the reaction rate Q, for which the problem is invariant and admits self-similar solutions.

Our attention is directed towards the so-called initiation problem of describing the flow from the initial time when the piston impacts, so the time when a steady-state solution takes effect. In recent years there has been much interest in experimentally measuring the flow parameters (particle velocity, pressure, specific volume, shock velocity etc.) in this regime, and numerical solutions have been extensively developed.

2. Formulation of the Model

We will use a Lagrangian description of the flow, with h denoting the Lagrangian position and t denoting time. Our convention is defined by the equation

$$dx = u\,dt + \frac{v}{v_0}\,dh \tag{1}$$

which relates the Eulerian position x to the Lagrangian position h. The quantities u and v, both functions of t and h, will denote particle velocity and specfic volume, respectively.

3. Basic Equations and Shock Conditions

The basic equation can be written as [19,21]:

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$$\frac{\partial v}{\partial t} - v_0(h)\frac{\partial u}{\partial h} = 0,$$
$$\frac{\partial u}{\partial t} + v_0(h)\frac{\partial p}{\partial h} = 0,$$
$$\frac{\partial p}{\partial t} + \frac{\gamma p}{v}\frac{\partial v}{\partial t} - \frac{(\gamma-1)q}{v}\frac{\partial \lambda}{\partial t} = 0,$$
$$\frac{\partial \lambda}{\partial t} = Q(u,p,v,\lambda), \quad (2)$$

where v is the specific volume, u the particle velocity and p the pressure; all are functions of t and h. The dimensionless quantity λ, also a function of t and h, will denote the progress variable (mass fraction of the product) of an irreversible chemical reaction involving a single reactant and a single product. The quantity Q is the reaction rate that depends on the states u, p, v and λ. At present, we do not assume any specific form for Q. The constant quantity q is the energy liberated per unit mass in the chemical reaction, and γ is the polytropic exponent which is same for both the reactant and the product.

Let the initial condition at time t = 0 be given by u = 0, $v = v_0(h)$ and $p = p_0$, where the initial specific volume $v_0(h)$ is a function of h, and $p_0 > 0$ is an appropriate constant. The Rankine-Hugoniot jump conditions for the strong shock, $x = \varphi(t)$, give conditions just behind the shock (see[18]) as

$$u_1 = \frac{2}{\gamma+1}D, \frac{v_1}{v_0} = \frac{\gamma-1}{\gamma+1},$$
$$p_1 = \frac{(\gamma+1)}{2v_0}u_1^2, \quad (3)$$

where u_1, v_1 and p_1 are the values of u, v and p, respectively, just behind the shock, and $D = d\varphi/dt$ is the shock velocity.

4. Similarity Analysis by Invariance Groups

In order to obtain the similarity solutions of the system of equations (2) we derive its symmetry group such that the system (2) is invariant under this group of transformations. The idea of the calculation is to find a one-parameter infinitesimal group of transformations (see,[14,15])

$$h* = h + \varepsilon H, t* = t + \varepsilon T, u* = u + \varepsilon U,$$
$$v* = v + \varepsilon V, p* = p + \varepsilon P, \lambda* = \lambda + \varepsilon \Lambda, \quad (4)$$

where the infinitesimals H, T, U, V, P and Λ are functions of t, h, u, p, v and λ. These infinitesimals are to be determined in such a way that the system (2), together with the jump conditions (3), is invariant under the group of transformations (4); the entity ε is a small parameter such that its square and higher powers may be neglected. The existence of such a group reduces the number of independent variables by one, which allows us to replace the system (2) of partial differential equations by a system of ordinary differential equations.

We introduce the notation $x_1 = h$, $x_2 = t$, $u_1 = u$, $u_2 = v$, $u_3 = p$, $u_4 = \lambda$ and $p_j^i = \partial u_i / \partial x_j$, where i = 1, 2, 3, 4 and j = 1, 2.

The system (1), which can be represented as

$$G_r(x_j, u_i, p_j^i) = 0, r = 1,2,3,4$$

is said to be constantly conformally invariant under the infinitesimal group of transformations (4) if there exist constants α_{rm} (r, m = 1, 2, 3, 4) such that

$$\mathcal{L}\, Gr = \alpha rm\, Gm, r, m = 1,2,3,4, \quad (5)$$

where $\mathcal{L}$ is the extended infinitesimal generator of the group of transformations (4), and is given by

$$\mathcal{L} = \xi^j \frac{\partial}{\partial x_j} + \eta^i \frac{\partial}{\partial u_i} + \beta_j^i \frac{\partial}{\partial p_j^i}, \quad (6)$$

where $\xi^1 = H$, $\xi^2 = T$, $\eta^1 = U$, $\eta^2 = V$, $\eta^3 = P$, $\eta^4 = \Lambda$ and

$$\beta_j^i = \frac{\partial \eta^i}{\partial x_j} + \frac{\partial \eta^i}{\partial u_k} p_j^k - \frac{\partial \xi^i}{\partial x_j} p_\ell^i - \frac{\partial \xi^l}{\partial u_n} p_\ell^i p_j^n, \quad (7)$$

where l = 1, 2, n = 1, 2, 3, 4, j = 1, 2, i = 1, 2, 3, 4 and k = 1, 2, 3, 4; here repeated indices imply summation convention.

Equation (5) implies

$$\xi^j \frac{\partial G_r}{\partial x_j} + \eta^i \frac{\partial G_r}{\partial u_i} + \beta_j^i \frac{\partial G_r}{\partial p_j^i} = \alpha_{rm} G_m, \quad (9)$$

where r, m = 1,2,3,4. Substitution of β_j^i from (7) into (8) yields an identity in p_j^k and $p_\ell^i p_j^n$; hence we equate to zero the coefficients of p_j^i and $p_\ell^i p_j^n$; to obtain a system of first-order linear partial differential equations in the infinitesimals H, T, U, V, P and Λ. This system, called the system of determining equations of the group of transformations, is solved to find the invariance group of transformations. We apply the above procedure to each equation of the system (1) and R-H conditions, and obtain the system of determining equations in H, T, U, V, P and Λ. We solve this system of determining equations to obtain

$$H = bh + d, T = at + c, U = (b - a)u,$$
$$V = (a + \alpha 11)v, P = (2b - 3a - \alpha 11)p,$$
$$\Lambda = 2(b - a)\lambda, \quad (9)$$

where a, b, c, d and α_{11} are the arbitrary constants. Thus, the infinitesimals of the invariant group of transformations are completely known.

Also, we find that the reaction rate Q has the following form:

$$Q = p^{\beta/k_1} F\left(\frac{\lambda^{k_1}}{p}, \frac{u^{2k_1}}{p}, \frac{v^{\frac{k_1}{k_2}}}{p}\right), \quad (10)$$

where

$$k_1 = \frac{2b-3a-\alpha_{11}}{2(b-a)},$$
$$k_2 = \frac{\alpha_{11}+a}{2(b-a)}, \beta = \frac{2b-3a}{2(b-a)}. \quad (11)$$

5. Self-Similar Solutions

We use the invariant surface conditions to determine the similarity variable and the similarity solution. In the present case these conditions for u, p, v and λ, respectively, are

$$T\frac{\partial u}{\partial t} + H\frac{\partial u}{\partial h} = U,$$
$$T\frac{\partial p}{\partial t} + H\frac{\partial p}{\partial h} = P,$$
$$T\frac{\partial v}{\partial t} + H\frac{\partial v}{\partial h} = V,$$
$$T\frac{\partial \lambda}{\partial t} + H\frac{\partial \lambda}{\partial h} = \Lambda. \quad (12)$$

The characteristic equations corresponding to the equation

(12) is

$$\frac{dt}{at+c} = \frac{dh}{bh+d} = \frac{du}{(b-a)u}. \tag{13}$$

One first integral yields the similarity variable as

$$s = \frac{c_4 h+1}{(c_3 t+1)^{c_2}}, \tag{14}$$

where

$$c_2 = \frac{b}{a}, c_3 = \frac{a}{c}, c_4 = \frac{b}{d}.$$

The second first integral gives

$$u(t,h) = (c_3 t + 1)^{c_2 - 1}\,\hat{u}(s). \tag{15}$$

In the same manner, the second, third and fourth equations in (12) upon integration yield

$$p(t,h) = \hat{p}(s)(c_3 t + 1)^{2c_2 - c_5 - 3},$$
$$v(t,h) = \hat{v}(s)(c_3 t + 1)^{c_5 + 1},$$
$$\lambda(t,h) = \hat{\lambda}(s)(c_3 t + 1)^{2(c_2 - 1)}, \tag{16}$$

where $\hat{u}(s), \hat{p}(s)$, $\hat{v}(s)$ and $\hat{\lambda}(s)$ are the functions of the similarity variable s.

By substituting the self-similar forms of the solutions u, p, v and λ from equation (16) into the system (2) of partial differential equations, we obtain the following system of ordinary differential equations in $\hat{u}$, $\hat{p}$, $\hat{v}$ and $\hat{\lambda}$:

$$c_3(c_5 + 1)\hat{v} - c_2 c_3 s\frac{d\hat{v}}{ds} - k\, c_4\, d^{(a+a_{11})/b}\; s^{(c_5+1)/c_2}\frac{d\hat{u}}{ds} = 0,$$
$$(c_2 - 1)c_3\hat{u} - c_2 c_3 s\frac{d\hat{u}}{ds} + k d^{(a+a_{11})/b}\frac{d\hat{p}}{ds} = 0,$$
$$(2c_2 - c_5 - 3)c_4\hat{p} - c_2 c_4 s\frac{d\hat{p}}{ds} + \gamma\frac{\hat{p}}{\hat{v}}\left((c_5 + 1)\hat{v} - c_2 s\frac{d\hat{v}}{ds}\right) - c_4(\gamma - 1)\frac{q}{\hat{v}}\left(2(c_2 - 1)\hat{\lambda} - c_2 s\frac{d\hat{\lambda}}{ds}\right) = 0,$$
$$2(c_2 - 1)\, c_3\hat{\lambda} - c_2 c_3 s\frac{d\hat{\lambda}}{ds} = \hat{p}^{\beta/k_1}\, F, \tag{17}$$

where the similarity variable s is acting as the independent variable.

The initial conditions are given at s = 1 by

$$\hat{u}(1) = \hat{p}(1) = \hat{v}(1) = 1, \hat{\lambda}(1) = 0 \tag{18}$$

In summary, then, the mathematical problem of determining self-similar solutions has been reduced to solving the system (17) of ordinary differential equations subject to the initial conditions (18).

Since the shock must be a similarity curve, and pass through t = 0, h = 0, it follows that at s = 1 shock starts, and hence the shock path is given by

$$c_4 h + 1 = (c_3 t + 1)^{c_2}, \tag{19}$$

and the shock velocity is

$$D = \frac{dh}{dt} = D_0\,(c_3 t + 1)^{c_2 - 1}, \tag{20}$$

where

$$D_0 = \frac{c_2 c_3}{c_4}$$

is the initial shock velocity. The invariance of the jump condition yields the form of the initial specific volume as

$$v_0(h) = k(bh + d)^{\mu}, \tag{21}$$

where k is a constant and

$$\mu = \frac{a + \alpha_{11}}{b}.$$

6. Conclusions

We consider the hydrodynamic medium in which a chemical reaction occurs. The reaction is initiated by a plane shock wave which is introduced into the medium at time t = 0 by a driving piston. The method of Lie group invariance is used to obtain a class of self-similar solutions for this problem.

The equation (16) provides the forms of the self-similar solutions for u, p, v and λ, respectively. By substituting these self-similar forms of the solutions u, p, v and λ, into the system (2) of partial differential equations, we obtain the system (17) of ordinary differential equations in $\hat{u}$, $\hat{p}$, $\hat{v}$ and $\hat{\lambda}$. This system together with the initial conditions (18) can be solved numerically.

The equation (10) provides the form of the reaction rate Q, and the equation (21) yields the form of the initial specific volume $v_0(h)$ such that the problem is invariant and admits self-similar solutions. Consequently, it follows that the initial specific volume must satisfy the power law.

Also, the shock path is found in the equation (19) and the shock velocity is obtained in the equation (20). For the case of uniform initial specific volume, all these results match well with the solutions obtained in [19].

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