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Deflection of Structures using Modified Betti's Theorem

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Abstract In this paper, Betti's theorem is modified by inclusion of constraint reactions in the set of externally applied loads. Based on this, a new methodology for calculating deflections of any structure is presented in this paper. The methodology has an advantage over the conventional methods due to the fact that deflections of a structure for any general loading and for different boundary conditions are calculated mostly by simple multiplications. It does not require the knowledge of writing internal force/ moment expressions. The methodology leads to a unique concept of reference structural element. Equation of deflected elastic curve of the reference element is utilized to obtain equation of deflected elastic line of the structure with any other type of loading and boundary conditions. Present methodology is fast and easy in its use especially when the structure is loaded by a series of point loads. Even for distributed loads a simple integration of a priory known polynomial integrands leads to the result. This methodology is applicable to any class of structure as long as a reference element is defined for that class. Here, this methodology is illustrated for beams.

Keywords Betti's Theorem, Reciprocal Theorem, Indeterminate Beams, Modified Betti's Theorem

1. Introduction

Various methods exist for calculation of deflection/ deformation of structures which include energy methods ([1-2]), variational principles ([3-4]) and finite element methods ([5-6]). Take the case of beams, in conventional methods deflection of a beam is obtained invariably by using Internal Bending Moment (IBM) expression. For example, in double integration method IBM expression is to be integrated twice followed by evaluation of constants ([7]), in Castigliano's method single integration of the square of IBM is to be carried out followed by partial derivative with respect to the desired load ([8]), in Unit Load Method one has to integrate the product of IBM expression for the given load and IBM expression due to unit load ([9]). One does not have to work with internal force/ moment expressions if Betti's theorem is used after modifying it by including support reactions in the set of applied loads.

Betti's theorem, given by Enrico Betti in 1872 relates two systems of loads acting on an elastic body. It states that for a linear elastic body subjected to two different sets of forces {P} and {Q}; work, W_{PQ}, done by {P} while going through the displacements, $\{\delta_{pq}\}$, due to {Q} is equal to work, W_{QP}, done by {Q} while going through the displacements, $\{\delta_{qp}\}$, due to {P}. This theorem has been used by several authors for solving different problems of structural mechanics. Using this theorem, closed form solution for the displacement of inclusion problems was given by[10], singular stress field in the neighbourhood of notches and corners in anisotropic media was derived by[11] and the theorem was applied to elastic infinite space bounded by rigid inclusions by[12]. Selected problems of the elastic half-space were successfully solved by using reciprocal theorem by[13] and Betti's reciprocal theorem was applied to deformed bodies with different constitutive relations by[14].

In this paper, Betti's theorem is modified by including reactions from supports as part of the applied loading and a new methodology based on modified Betti's theorem is presented for getting deflection of any structure without using internal force/ moment expressions. It is much easier and faster as work done by external loading is calculated by multiplication of given loads acting on a given structure with the corresponding known displacements due to a chosen load vector on the structure and no integrations are involved if concentrated loads are acting on the given structure. Even for distributed loads integration of the product of two a priory known polynomial integrants (expression for the given distributed load and equation of deflection curve due to chosen load) is easy.

This methodology leads to a unique concept of reference structural element on which a conveniently chosen set of self equilibrating load is acting. With the help of the equation of its deflection curve, solution for deflection of the structure with different boundary conditions and for any other type of loading can be obtained. As the reference element remains same for a class of structures, this paves way for the development of a general purpose interactive graphic computer package for calculating deflections of a structure (belonging to the same class) with any general type of loads as well as with different boundary conditions.

Betti's theorem may be considered as a highly restricted

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form of principle of quasi work in which two systems are taken as topologically identical[15 – 19]. Whereas, modified Betti's theorem can be regarded as a restricted form of principal of quasi work in which two structural systems are taken to be topologically equivalent systems[15 – 19].

In the next section, the modified Betti's theorem is proved. In the subsequent sections the new methodology is developed and illustrated for beams.

2. Proof of Modified Betti's Theorem

Let two self equilibrating set of forces {P} and {Q} act on a structural system. In these sets some of the forces can be considered as reactions. This widens the scope of Betti's theorem as a structure with different boundary conditions along with different set of loads can be related.

As strain energy stored in the system does not dependent on the order in which loads are applied to the structure. There are two possible ways of applying self equilibrating set of loads {P} and {Q} on the structure. In the first case, load set {Q} is applied after the application of the load set {P}. In this case, Strain energy, U_1, stored in the structure is equal to:

$$U_1 = (1/2)\ \{P\}^T\{\delta\}_{PP} + \{P\}^T\{\delta\}_{PQ} + (1/2)\ \{Q\}^T\{\delta\}_{QQ} \quad (1)$$

Where, $\{\delta\}_{AB}$ = Deflection under loads 'A' due to loads 'B'

In the second case, load set {P} is applied after applying the load set {Q}. In this case, strain energy, U_2, stored in the structure is equal to:

$$U_2 = (1/2)\ \{Q\}^T\{\delta\}_{QQ} + \{Q\}^T\{\delta\}_{QP} + (1/2)\ \{P\}^T\{\delta\}_{PP} \quad (2)$$

Both U_1 and U_2 are the strain energy stored in the structural system under the application of the two sets of self equilibrating loads ({P} and {Q}). As $U_1 = U_2$, Eqn.(1and 2) yield:

$$\{P\}^T\ \{\delta\}_{PQ} = \{Q\}^T\ \{\delta\}_{QP}. \quad (3)$$

This can be states in words as:

Work, W_{PQ}, done by the self equilibrating set of forces, {P}, while going through the displacements, $\{\delta\}_{PQ}$, due to self equilibrating set of forces, {Q}, is equal to work, W_{QP}, done by self equilibrating set of forces, {Q}, while going through the displacements, $\{\delta\}_{QP}$, due to self equilibrating set of forces, {P}.

$$\text{OR } W_{PQ} = W_{QP} \quad (4)$$

3. Reference Structure

As this methodology is based on modified Betti's theorem (MBT), therefore, two sets of loads for the same structure should be given. One set of loads acting on a given structure is predefined and other set is to be chosen by analyst. This chosen set of self equilibrating loads acting on the structure should remain same for all given problems. It is of interest to observe that the structure in MBT does not include supports. Hence, both loading and support conditions have to be defined for the reference structure. Here, this concept is illustrated for beams and an attempt has been made to define such a beam. This beam will be referred to as Reference Beam (RB).

A simply supported beam 'CD' carrying a point load 'P_2' acting downwards at a distance of ζ from the left end 'C', shown in Fig.1, is taken as RB. Its material and geometrical properties will be same as that of the given beam. Let these properties be E, I and L.

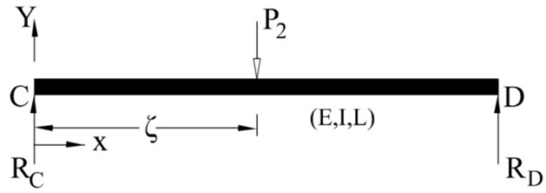

Figure 1. Reference Beam

The loads and corresponding deflections of this beam will be denoted by subscript '2'. Equation of its deflected elastic curve is given by:

$$v_2(x) = \frac{P_2}{6EIL}\{(L-\zeta)x(x^2+\zeta^2-2L\zeta) - L<x-\zeta>^3\} \quad (5)$$

4. Application to Beams

In what follows are illustrative examples of the application of present methodology for obtaining equation of deflected elastic curve of beams and also for calculating the deflection at a point in a given beam. Examples include determinate and indeterminate beams with different boundary conditions carrying point as well as distributed loads. In these illustrations loads and corresponding deflections of the given beam will be denoted by subscript '1'.

4.1. Illustration – 1: S. S. Beam Partly Uniformly Loaded

Equation for deflected elastic line of a simply supported beam 'AB' carrying a uniform load of 'w_1' N/m as shown in Fig.2 is derived. The load acts from 'a' meters to '(a + b)' meters as measured from the left end.

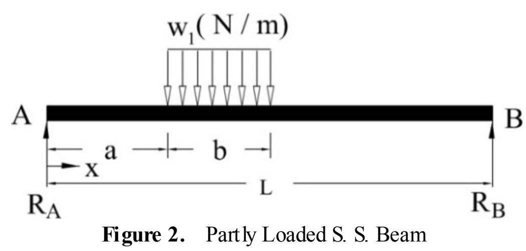

Figure 2. Partly Loaded S. S. Beam

Work done by the given distributed load and support reactions while going through the deflections of RB, Eqn.(5), is equal to:

$$W_{12} = \int_a^b (-w_1)v_2(x)dx + R_A v_2(0) + R_B v_2(L)$$

$$W_{12} = -P_2 w_1[(L-\zeta)\{(b^4-a^4) + 2\zeta(\zeta-2L)(b^2-a^2)\} - L(<b-\zeta>^4 - <a-\zeta>^4)\ ]/24EIL \quad (6)$$

Work, W_{21}, done by load 'P_2' of RB while going through the deflection, $v_1(\zeta)$, caused by the loads of the given beam is equal to $-P_2\ v_1(\zeta)$. Using $W_{12} = W_{21}$, the deflection of given beam at any arbitrary point 'ζ' is given by:

$$v_1(\zeta) = w_1[(L-\zeta)\ \{(b^4 - a^4) + 2(\zeta^2 - 2L\zeta)(b^2 - a^2)\} \\ - L\ \{(<b-\zeta>^4 - <a-\zeta>^4\}\ ]/24EIL \quad (7)$$

As 'ζ' is arbitrary it can as well be replaced by 'x' thereby giving Eqn.(7) the conventional form. For a uniformly distributed load over the full span (a = 0 and b = L), the above equation reduces to

$$v_1(x) = -wx(x^3 - 2Lx^2 + L^3)/24EI \quad (8)$$

Which is same as given in[9] page 598.

4.2. Illustration – 2: S. S. Beam with Point Loads

In this example one can observe that for point loads only evaluation of algebraic expression (summation of the product of two numbers) is required which definitely is much simpler than what existing conventional methods offer.

A simply supported beam 'AB' of length 8m carrying two point loads and an end moment as shown in Fig.3 is taken up for calculating deflection under the load of 64kN acting downwards at a distance of 1m from the left end 'A' of the beam. Other loads acting on the beam are taken as a point load of 48kN acting downwards at a distance of 4m from end 'A' and a clockwise moment of 10kN.m acting at the right end 'B' of the beam. The value of young's modulus of elasticity, E= 210 GPa and second moment of cross sectional area, I = 180 x 10^6 mm^4.

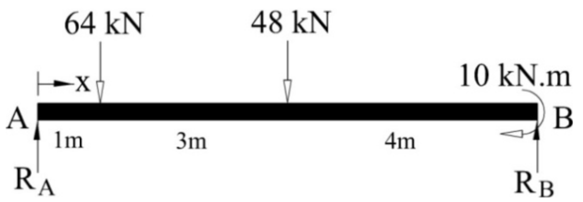

Figure 3. S. S. Beam with Point Loads

Using Eqn.(5) with ζ = 1 (point where deflection is to be calculated in the given beam), W_{12} is calculated as follow:

$$W_{12} = R_A v_2(0) - 64v_2(1) - 48v_2(4) - 10v_2'(8) + R_B v_2(8) \\ = -64(-2.041\bar{6}P_2) - 48(-3.91\bar{6}P_2) - 10(1.3125)P_2 \\ = 305.541\bar{6}P_2/EI \quad (9)$$

Using MBT, W_{12} = W_{21}[= $-P_2$ v_1 (1)], deflection of the given beam at a distance of ζ = 1m from the left end is equal to v_1 (1) = - 305.5417/EI = – 8.083 mm.

4.3. Illustration – 3: Deflection of a Beam with Overhang

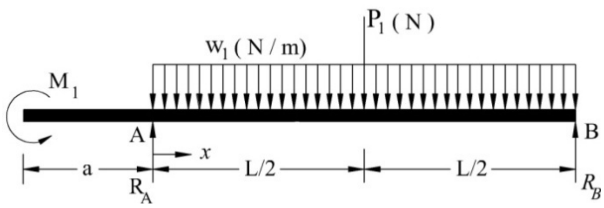

Figure 4. Beam with overhang

To obtain equation of deflected neutral axis between the supports of a simply supported beam 'AB' of length 'L' with an overhang of length 'a' to the left of the support 'A' having parameters 'EI', as shown in Fig.4, is chosen as the given problem. The loading on it is taken as an anticlockwise moment M_1 acting at the free end, P_1 acting at L/2 from the right end 'B' and a uniform load of intensity w_1 N/m between the supports. Let its other parameters be E and I.

As the equation for neutral axis between the supports is required, the end moment M_1 is statically transferred to support 'A'. Using Eqn.(5) of RB, W_{21} is calculated as follows:

$$W_{12} = M_1 v_2'(0) + R_A v_2(0) - P_1 v_2(L/2) \\ + R_B v_2(L) + \int_0^L (-w_1) v_2(x) dx$$

$$W_{12} = \frac{P_2}{48EI}[2(L-\zeta)\ \zeta\ \{4M_1(\zeta - 2L) + Lw_1(L^2 + L\zeta - \zeta^2)\} \\ - P_1\{(L^3 - 9L^2\zeta + 12L\zeta^2 - 4\zeta^3) - <L-2\zeta>^3\}] \quad (10)$$

W_{21} = $-P_2$ $v_1(\zeta)$. Applying MBT, equation for deflection, $v_1(\zeta)$, of the neutral axis of the given beam is given by:

$$v_1(\zeta) = [P_1\{(L^3 - 9L^2\zeta + 12L\zeta^2 - 4\zeta^3) - <L-2\zeta>^3\} \\ -2\zeta(L-\zeta)\{4M_1(\zeta - 2L) + Lw_1(L^2 + L\zeta - \zeta^2)\}]/48EI \quad (11)$$

If M_1 = w_1 $L^2/12$ and P_1 = 0 then Eqn.(11) simplifies to:

$$v_1(\zeta) = -w_1\zeta(3\zeta^3 - 7L\zeta^2 + 3L^2\zeta + L^3)/72EI \quad (12)$$

Replacing ζ by x in the above equation yields the equation for deflection in conventional form which is same as given in[20].

4.4. Illustration – 4: Cantilever Beam with Uniform Load

In this illustration deflection equation of elastic curve of a cantilever beam, 'AB' shown in Fig.5, with parameters E, I, L and carrying a uniform load, 'w' N/m, over its full span is obtained.

W_{12} and W_{21} are given by:

$$W_{12} = P_2 w_1(3L^3\zeta - 6L^2\zeta^2 + 4L\zeta^3 - \zeta^4)/24EI \quad (13)$$

$$W_{21} = R_C v_1(0) + P_2 v_1(\zeta) + R_D v_1(L) \\ = P_2 v_1(\zeta) - (P_2\zeta/L)v_1(L) \quad (14)$$

Using MBT (W_{12} = W_{21}) yields following equation

$$v_1(\zeta) - (\zeta/L)v_1(L) = \frac{w_1\zeta}{24EI}(3L^3 - 6L^2\zeta + 4L\zeta^2 - \zeta^3) \quad (15)$$

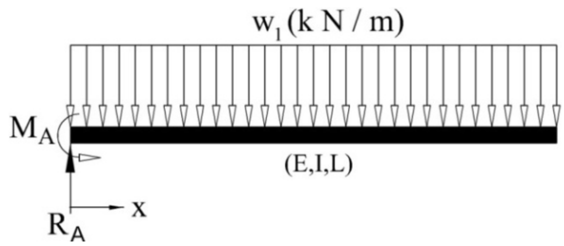

Figure 5. Cantilever Beam with uniform load

Slope boundary condition at end 'A' of the given cantilever yields deflection at the free end, v_1 (L) = $-w_1$ L^4/ 8EI. With this value of tip deflection, Eq.(15) yields the equation of the given deflected beam as:

$$v_1(\zeta) = -w_1\zeta^2(\zeta^2 - 4L\zeta + 6L^2)\}/24EI \quad (16)$$

Replacing ζ by (L-x), Eqn.(16) reduces to the conventional form as given in[9] p853.

4.5. Illustration - 5: Deflection of Single Degree Indeterminate Beam

In this illustration, equation for deflected neutral axis of indeterminate beam 'AB' supported by a roller at the left end 'A' and built-in at right end 'B' is obtained. Beam has parameters L, E, I and carries an anticlockwise moment 'M_1' at its midpoint as shown in Fig.6 wherein R_A, R_B and M_B are reactions from supports. Parameters of RB in Eqn.(5) are chosen to be same as those of the given beam parameters (E, I, L).

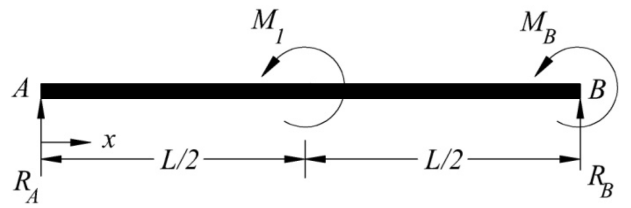

Figure 6. Tip Supported Cantilever Beam

Quasi work W_{12} is given by:

$$W_{12} = -\frac{P_2}{6EIL}[M_1\{3L<L/2-\zeta>^2 -(L-\zeta)(3L^2/4+\zeta^2-2L\zeta)\} - M_B(L-\zeta)\zeta(L+\zeta)] \tag{17}$$

W_{21} = -P $v_1(\zeta)$. Applying MBT, the equation for deflection, $v_1(\zeta)$, of the neutral axis of the given beam is given by:

$$v_1(\zeta) = \frac{1}{6EIL}[M_1\{3L<L/2-\zeta>^2 -(L-\zeta)(3L^2/4+\zeta^2-2L\zeta)\} - M_B(L-\zeta)\zeta(L+\zeta)] \tag{18}$$

Using boundary condition that slope at ζ = L is zero, yields value of redundant moment reaction M_B as M/8. With this value of M_B, equation for the deflection becomes:

$$v_1(\zeta) = M_1[9\zeta^3/L - 3L\zeta + 24<\zeta - L/2>^2]/48EI \tag{19}$$

Replacing ζ by x and substituting 2L for L yields the same equation as given in[20].

4.6. Illustration – 6: Two Degree Indeterminate Beam

Equation of the deflected elastic line of a beam 'AB' built in at both ends, as shown in Fig.7, having parameters E, I, L and uniformly loaded from a distance 'a' to a distance 'b' from the left end 'A' is obtained in this illustration.

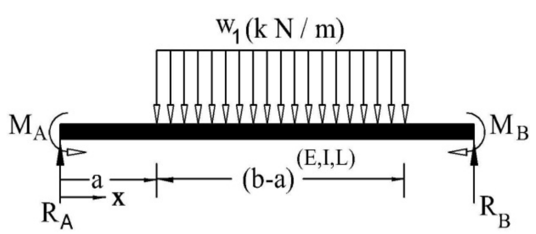

Figure 7. Built - in Beam with UDL

The given beam is two degree indeterminate. Using Eqn.(5), W_{12} is given by:

$$W_{12} = \frac{P_2}{24EIL}[w_1\{(L-\zeta)(b^2-a^2)(a^2+b^2+2\zeta^2-4L\zeta) - L(<b-\zeta> - <a-\zeta>)\} + 4M_A(L-\zeta)\zeta(\zeta-2L) - 4M_B(L-\zeta)\zeta(\zeta-5L)] \tag{20}$$

W_{21} = -P $v_1(\zeta)$. Applying MBT, equation for deflection, $v_1(\zeta)$, of the neutral axis of given beam is given by:

$$v_1(\zeta) = -[w_1\{(L-\zeta)(b^2-a^2)(a^2+b^2+2\zeta^2-4L\zeta) - L(<b-\zeta> - <a-\zeta>)\} + 4M_A(L-\zeta)\zeta(\zeta-2L) - 4M_B(L-\zeta)\zeta(\zeta-5L)]/24EIL \tag{21}$$

The values of M_A and M_B are obtained from the solution of two equation obtained by applying slope boundary conditions at the ends 'A' (ζ = 0) and 'B'(ζ = L) and are give by:

$$M_A = w_1[4L(b^3-a^3)-3(b^4-a^4)]/12L^2 \quad \text{and} \quad M_B = w_1[16L(b^3-a^3)-9(b^4-a^4)-6L^2(b^2-a^2)]/12L^2 \tag{22}$$

For a beam uniformly loaded throughout its span, the equation for its deflection curve is obtained by substituting a = 0, b = L and the corresponding values of M_A and M_B from Eqn.(22) in Eqn.(21), which then reduces to:

$$v_1 = -w_1\zeta^2(L-\zeta)^2/24EI \tag{23}$$

5. Conclusions

1) Betti's theorem is successfully modified by treating reactions from constraints as a part of external loading.

2) Using modified Betti's theorem, deflection at any point of a given beam or the equation of its deflected elastic line is successfully obtained without using internal bending moment expressions.

3) A reference beam is defined by choosing its loading and boundary conditions. Equation for its deflected elastic line is successfully used to solve all other beam problems irrespective of their loading and boundary conditions.

4) Obtaining deflection at a point or equation of deflected elastic line of any given beam becomes easy and simple by using present methodology.

5) It is further simplified if given problem has only point loads as the solution is obtained by simple multiplications. In this case neither integration nor differentiation is required.

6) As reference beams is used to solve all other beam problems with different loads and different constraints, it is possible to develop an interactive graphic general purpose computer package for this purpose.

7) This method is a general one and can be used to solve any other class of structural problems as long as a reference structural element for that class of structures is defined.

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