Deflection of Structures Using Principle of Quasi Work

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Abstract   A new methodology for obtaining deflections of structures is presented in this paper. It is based on Principle of Quasi Work, which connects two topologically similar systems thereby leading to a unique concept of standard elements for every category of structural problems. Using a priory known equation of deformation/deflection of elastic axis of these elements, solution for equation of deformed/ deflected elastic line of other structural members of similar category having different loading and boundary conditions is presented here. This methodology is easy to use as deflection of a given structure is obtained mostly by simple multiplications and it also eliminates the use of internal force/bending moment expressions unlike conventional methods. Even though this methodology can be applied to any structure, its use is illustrated for one dimensional structural elements (viz. axial bars, torsion rods and beams) for the sake of conciseness and clarity. The concept of topologically similar system is explored for each category of elements as it lies at the heart of the principle of quasi work.

Keywords   Energy Methods, Structural Analysis, Beams, PQW, TSS, TSB, TEB

1. Introduction

Successful attempts of transporting structural mechanics problems into other fields of science have been made by various authors. Structural mechanics problems were solved using graph theory by[1, 2]; through electrical networks by[3]; and resorting to combination of graph theory and electric networks by[4]. In all these cases, one needs knowledge of other fields of science in addition to the knowledge of structural mechanics. To overcome this crippling disadvantage Principle of Quasi Work (PQW) in the realm of structural mechanics was derived by[5] by importing Tellegen's theorem as given in[6] thereby making knowledge of other fields of science for any user redundant. PQW establishes connection between two Topologically Similar Systems, (TSS), in the field of structural mechanics. All the presently available structural analysis procedures e.g. finite element methods[7, 8, 9, 10]; variational principles[11, 12] and energy methods[13, 14] meant for a single structure do not provide any connection between two structural systems. PQW thus fills existing void in the domain of structural mechanics. Using PQW, some useful theorems for discrete structural models were derived by[5, 15]. PQW was advantageously used by[16] for obtaining redundant reactions of beams. A quick and simplified method for obtaining nodal deflections of an indeterminate truss using PQW was given by[17]. PQW has much wider applicability as all the existing energy principles

can be derived as special cases of PQW by restricting TSS to topologically identical systems i.e. when two systems are identical with each other in every respect. PQW utilizes known solution of a TSS to solve other determinate/ indeterminate topologically similar problems.

All the available methods for calculation of deflection are based on internal force (/ torque/ bending moment/ etc.) distributions. PQW gives a choice of either using internal force expressions or completely bypassing these. First case (of using these distributions) is a direct application of PQW ($W_{mn}=U_{mn}$). In the latter case (of bypassing these distributions), solution is obtained by restricting TSS to topologically equivalent systems (TES). For TES $U_{mn}=U_{nm}$ and hence, PQW can be used in the form of $W_{mn}=W_{nm}$. Working with quasi work expressions is much easier compared to quasi strain energy expressions. Quasi work (W_{mn}) is calculated by multiplying applied loads from system 'm' with corresponding known displacements of system 'n' and no integrations are involved if only concentrated loads are acting on the given system. Even for distributed loads the integration is easy as the integrants (the expression for distributed load acting on the given system, TES_m, and the equation for the deflection curve of chosen system TES_n) are predefined. PQW involves two TSS out of which one represents the given determinate/ indeterminate problem at hand and other TSS with a priory known equation of its deformed/ deflected elastic line is chosen by the user. Hence, a judicious choice of TSS_2 leads to a unique standard element for every category of problem.

Deformation/ deflection equations of these standard elements, used for obtaining the deformation/ deflection of other given determinate/ indeterminate problems (of the same class) under different loading and boundary conditions,

remain same thereby paving way for development of a general purpose interactive computer program for obtaining the deflection of these structural members.

In this paper, conditions which axial bars and torsion rods have to satisfy for topological similarity are looked into. PQW is validated for these elements. An attempt is made to define standard element (in each case) whose solution for deflection/ deformation forms the basis for obtaining deflections/ deformations of other given problems. Few illustrative examples are also included.

2. Principle of Quasi Work

Principle of quasi work was first proved by Panditta[5] and subsequently stated by Panditta et.al.[15,16]. According to this principle quasi work, 'W$_{mn}$', done by external forces of one system (say 'm') while going through the corresponding displacements of the other topologically similar system (say 'n') is equal to the quasi strain energy, 'U$_{mn}$', due to internal forces of former system ('m') while going through corresponding deformations of the latter system ('n'). In mathematical terms it can be stated as:

$$W_{mn} = U_{mn} \qquad (1)$$

Where, U$_{mn}$ = Quasi strain energy = $\{F\}_m^T \{\delta\}_n$; W$_{mn}$ = Quasi work = $\{P\}_m^T \{d\}_n$; $\{P\}_m$ = External loads acting on TSS$_m$; $\{d\}_n$ = Displacement in TSS$_n$ corresponding to $\{P\}_m$; $\{F\}_m$ = Internal forces in TSS$_m$; $\{\delta\}_n$ = Deformation in TSS$_n$ corresponding to $\{F\}_m$ and $\{(exp.)\}_i$ = (expr.) is for TSS$_i$.

In this principle, word 'corresponding' needs clarification. For example, take two beams 'AB' and 'CD' having lengths 'L$_1$' and 'L$_2$'. In these two beams end 'A' corresponds to end 'C' and end 'B' to end 'D'. In such cases, one has to resort to mapping in the two structural domains. For example, if a transverse load is acting at 0.75L$_1$ from end 'A' in beam 'AB' then its corresponding point in beam 'CD' will be 0.75L$_2$ from end 'C'.

2.1. Topological Similarity of Structural Elements

In this section, condition(s) which axial bars and torsion rods should satisfy for being topologically similar will be arrived at. These conditions are deduced from calculations of quasi energy associated with the two structural elements. This also sets the conditions for the applicability of PQW as quasi energy is a part of PQW.

2.1.1. Topological Similar Axial Bars

Let two axial bars with parameters L$_m$, E$_m$, A$_m$ and L$_n$, E$_n$, A$_n$ be represented by TSS$_m$ and TSS$_n$, respectively. Quasi energy, 'U$_{mn}$' is written as follows:

$$U_{mn} = \int_{A_m} \int_0^{L_n} \{\sigma_m(L_n x_m / L_m) dA_m\}_m \{\varepsilon_n(x_n) dx_n\}_n \quad (2)$$

Where, σ(x) = Normal stress as a function of 'x'; ε(x) =

Normal Strain as a function of 'x'; A = Cross section area; L= Length of structural member; m, n = Subscripts representing different TSS and 'x' is the coordinate along longitudinal axis.

As length of the two bars is different, mapping of x–coordinate from TSS$_m$ to TSS$_n$ gives x$_m$ L$_n$/L$_m$ and x$_n$ as corresponding points in these TSS. In such a situation it is better to work in terms of non- dimensional parameter ξ = x$_m$ / L$_m$= x$_n$ / L$_n$. Eqn.(2) in non – dimensional variable becomes:

$$U_{mn} = \int_{A_m} \int_0^1 \{\sigma_m(L_n\xi) dA_m\}_m \{\varepsilon_n(L_n\xi)\, L_n d\xi\}_n$$

$$= \int_{A_m} \int_0^1 \{\frac{F_m(L_n\xi)}{A_m} dA_m\}_m \{\frac{F_n(L_n\xi)}{A_n E_n} L_n d\xi\}_n$$

$$= \int_0^1 \frac{L_n}{A_n E_n} F_m(L_n\xi) F_n(L_n\xi)\, d\xi \qquad (3)$$

Similarly

$$U_{nm} = \int_0^1 \frac{L_m}{A_m E_m} F_m(L_m\xi) F_n(L_m\xi)\, d\xi$$

Where, E = Young's modulus of elasticity. If mapping is done from TSS$_n$ to TSS$_m$ then quasi energy is given by:

$$U_{mn} = \int_0^1 \frac{L_m}{A_n E_n} F_m(L_m\xi) F_n(L_m\xi)\, d\xi \quad \text{and}$$

$$U_{nm} = \int_0^1 \frac{L_n}{A_m E_m} F_m(L_n\xi) F_n(L_n\xi)\, d\xi \qquad (4)$$

For deriving Eqn.(3) no restrictions on any geometric parameters of the two bars was required. Hence, two bars having different geometric parameters are topologically similar. It paves way for the application of PQW. If topologically similar axial bars (TSAB) also have same distribution of the parameter AE (A$_m$E$_m$ = A$_n$E$_n$) along their length then these axial elements will be designated as Topologically Equivalent Axial Bars (TEAB). For these elements quasi strain energy U$_{mn}$ given by Eqn.(3), simplifies to:

$$U_{mn} = \int_0^1 \frac{L_n}{AE} F_m(L_n\xi) F_n(L_n\xi)\, d\xi \qquad (5)$$

For TEAB, Eqns. (4 and 5) give U$_{mn}$ = U$_{nm}$ and along with Eqn.(1) offer six equations (viz. W$_{mn}$ = U$_{mn}$ = U$_{nm}$ = W$_{nm}$) for their analysis instead of usual two equations W$_{mn}$ = U$_{mn}$ and W$_{nm}$ = U$_{nm}$. If the length of the TEAB is also equal then the Eqn.(5) is written as:

$$U_{mn} = \int_0^L \frac{F_m(x) F_n(x)}{AE}\, dx \qquad (6)$$

Illustration – 1: Validation for Axial Bars

Consider two axial bars with parameters A$_1$, E$_1$, L$_1$ and

A_2, E_2, L_2. Let the ends of first member be fixed – fixed (Fig.1a) and that of second be fixed – free (Fig.1b).
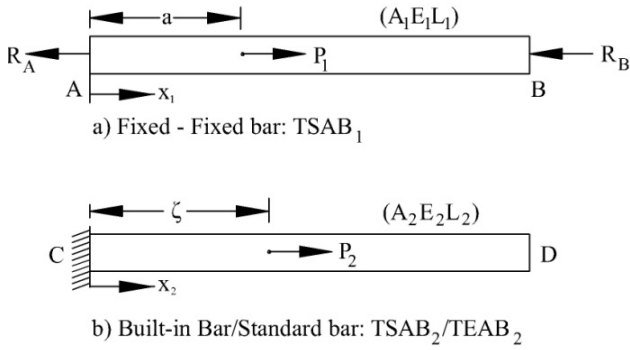

a) Fixed - Fixed bar: $TSAB_1$

b) Built-in Bar/Standard bar: $TSAB_2/TEAB_2$

Figure 1. Topologically Similar Bars

Loading on first be a force 'P_1' Newton acting at a distance of 'a' meters and that of the other be 'P_2' Newton acting at a distance of 'ζ' meters measured from left end in both the cases. Let R_A and R_B be the support reactions. First one will be designated by $TSAB_1$ and other by $TSAB_2$. Internal force distribution, $F(\xi)$, and axial deformation, $u(\xi)$, in terms of non-dimensional parameter 'ξ' for $TSAB_1$ and $TSAB_2$ are given hereunder:

$$\{F_1(\xi)\}_1 = R_A - P_1 < \xi - a/L_1 >^0$$
$$\{u_1(\xi)\}_1 = L_1(R_A\xi - P_1 < \xi - a/L_1 >)/A_1E_1 \quad (7)$$

$$\{F_2(\xi)\}_2 = P_2 - P_2 < \xi - \zeta/L_2 >^0$$
$$\{u_2(\xi)\}_2 = L_2P_2(\xi - < \xi - \zeta/L_2 >)/A_2E_2 \quad (8)$$

Where, $< (\exp.) > = 0$ if $(\exp.) \leq 0$ otherwise it is equal to $(\exp.)$. Using Eqn.(3)

$$U_{12} = \{-R_B\zeta + P_1(\zeta - < \zeta - a >)\}_1\{P_2/A_2E_2\}_2 \quad (9)$$
and

$$
\begin{aligned}
W_{12} &= \{-R_A\}_1\{u_2(0)\}_2 + \{P_1\}_1\{u_2(a/L_1)\}_2 \\
&\quad + \{-R_B\}_1\{u_2(1)\}_2 \\
&= 0 + [\{P_1\}_1\{a/L_1 - < a/L_1 - \zeta/L_2 >\}_2 \\
&\quad - \{R_B\}_1\{(1 - < 1 - \zeta/L_2 >)\}_2]\{P_2L_2/A_2E_2\}_2 \\
&= [-\{R_B\}_1\{\zeta\}_2 + \{P_1\}_1\{\zeta - < \zeta - aL_2/L_1 >\}_2] \\
&\quad \{P_2/A_2E_2\}_2
\end{aligned} \quad (10)
$$

Referring displacements to $TSAE_1$, Eqn.(10) can be written as Eqn.(11) or Eqn.(12).

$$W_{12} = \{-R_B\zeta + P_1(\zeta - < \zeta - a >)\}_1\{P_2/A_2E_2\}_2 \quad (11)$$

$$W_{12} = \{-R_B\zeta + P_1(a - < a - \zeta >)\}_1\{P_2/A_2E_2\}_2 \quad (12)$$

From Eqns.(9,11) PQW stands validated. Similarly one can show that $U_{21} = W_{21}$.

2.1.2. Topological Similar Shafts

Representing two shafts with circular cross section and having parameters L_m, G_m, J_m and L_n, G_n, J_n by TSS_m and TSS_n, the expression for U_{mn} derived on similar lines as for axial elements is given below:

$$U_{mn} = \int_{A_m}\int_0^{L_n} \{\tau_m(L_n x_m/L_m)dA_m\}_m \{\gamma_n(x_n)\,dx_n\}_n \quad (13)$$

In non –dimensional form, U_{mn} is given by:

$$
\begin{aligned}
U_{mn} &= \int_{A_m}\int_0^1 \{\tau_m(L_n\xi)dA_m\}_m \{\gamma_n(L_n\xi)\,L_n d\xi\}_n \\
&= \int_{A_m}\int_0^1 \{\frac{T_m(L_n\xi)r_m}{J_m}dA_m\}_m \{\frac{T_n(L_n\xi)r_n}{J_nG_n}L_n d\xi\}_n \quad (14) \\
&= \int_0^1 L_n\left\{\frac{T_m(L_n\xi)}{J_m}\right\}_m\left\{\frac{T_n(L_n\xi)}{G_nJ_n}\right\}_n d\xi \int_{A_m} r_m r_n dA_m
\end{aligned}
$$

Where, $\tau(x)$ = Shear stress as a function of 'x'; $\gamma(x)$= Shear strain as a function of 'x'; J = Polar moment of cross sectional area; G =Shear modulus of elasticity and $T(x)$ is applied torque as a function of x.

For meaningful evaluation of the integral over A_m, it will be prudent to take two shafts of equal diameter. With this corresponding points in radial direction on the two shafts will be at the same radial distance from the shaft centres (i.e. $r_m = r_n$) and $J_m = J_n$ (= J). The second integral evaluates to J_m (=J). With this, expression for U_{mn} in Eqn. (14) reduces to:

$$U_{mn} = \int_0^1 \frac{L_n}{G_nJ}T_m(L_n\xi)T_n(L_n\xi)\,d\xi \quad (15)$$

If mapping is done from TSS_n to TSS_m, then U_{mn} takes the form:

$$U_{mn} = \int_0^1 \frac{L_m}{G_nJ}T_m(L_m\xi)T_n(L_m\xi)\,d\xi \quad (16)$$

Derivation of Eqn. (15) necessitates that the radius of two shafts should be equal. Hence, for shafts to be topologically similar their diameter should be equal. For topologically equivalent rods (TER) with $G_m = G_n = G$ Eqn. (15, 16) gives $(W_{mn}=) U_{mn} = U_{nm} (=W_{nm})$.
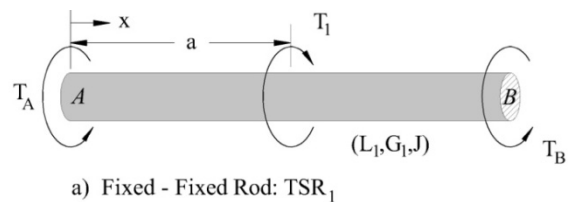
Illustration – 2: Validation for Shafts

a) Fixed - Fixed Rod: TSR_1

b) Built-in Rod/Standard Rod: TSR_2/TER_2

Figure 2. Topologically Similar Rods

Consider a fixed - fixed torsion rod with circular section and having parameters J, G_1, L_1 and an anticlockwise torque 'T_1' Nm acting at a distance of 'a' meters from left end 'A' as shown in (Fig. 2a). This rod will be designated by TSS_1. Topologically similar rod, TSS_2, will be taken as a fixed - free torsion rod with circular section having parameters J, G_2, L_2 and an anti - clockwise torque 'T_2' N.m acting at a distance of 'ζ' meters from left end 'C' as shown in (Fig. 2b).

The non – dimensional internal torque, T(x), and angle of twist, θ(x), distributions in TSR_1 and TSR2 are given by:

$$\{T_1(\xi)\} = T_A - T_1 < \xi - a / L_1 >^0$$
$$\{\theta_1(\xi)\} = L_1(T_A\xi - T_1 < \xi - a / L_1 >) / G_1 J \quad \text{and}$$
$$\{T_2(\xi)\} = T_2 - T_2 < \xi - \zeta / L_2 >^0 \qquad (17)$$
$$\{\theta_2(\xi)\} = L_2(T_2\xi - T_2 < \xi - \zeta / L_2 >) / G_2 J$$

U_{12} and W_{12} are given by:

$$U_{12} = T_2[-T_B\zeta + T_1(\zeta - < \zeta - a >] / G_2 J \quad (18)$$
$$W_{12} = T_2[-T_B\zeta + T_1(\zeta - < \zeta - a) / G_2 J \quad (19)$$

From Eqns.(18,19) PQW stands validated. Same is true for U_{21} and W_{21}.

2.1.3. Topological Similar Beams

Condition for topologically similar beams has been derived by Panditta et.al.[16] and it requires that the two beams should have equal depth. Quasi strain energy, U_{mn}, of two topologically similar beams (TSB) with parameters (L, A_m, E_m) and (L, A_n, E_n), referred to as TSB_m and TSB_n, is given by:

$$U_{mn} = \int_0^L \frac{1}{E_n I_n} \{M_m(x)\}_m \{M_n(x)\}_n \, dx \qquad (20)$$

For topologically equivalent beams (TEB) with $E_m I_m = E_n I_n = EI$, Eqn. (20) along with Eqn. (1) as usual yield six equations for their analysis.

3. Standard Elements

After validating PQW for one dimensional elements it can be observed that PQW relates two structural systems through quasi wok and quasi strain energy. Hence, it is possible to use known solution of appropriately predefined TSS_2 to get solution of all other determinate/ indeterminate problems (TSS_1) of the same class. In other words, one can utilize expression for internal force (for calculating U_{mn}) and deformation/ deflections (for obtaining W_{mn}) of one unique TSS (of a class) and utilize it for obtaining solutions of other TSS (of the same class). This is of particular interest in structural engineering as one has to deal with indeterminate structures whose solution can now be obtained by the solution of a determinate structure (predefined TSS_2) of the same class. Moreover, there is a possibility of utilizing exclusively quasi work expressions (which do not need internal force distribution) by choosing two systems to be

TES which necessitates that the two structural systems should have same stiffness parameter (AE/ GJ/ EI/ etc.). Here an attempt is made to define standard elements for axial bars, torsion rods and beam elements. These elements will be referred to as standard elements.

Axial member shown in Fig. 1b will be used as a standard bar. It will be designated as $TSAB_2$/ $TEAB_2$. Cantilever shaft with circular section shown in Fig.2b will be used as a standard torsion rod. It will be designated as TSR_2/ TER_2.

Standard beam for deflection (SBD), TSB_2 / TEB_2, is chosen as a simply supported beam 'CD' with parameters L, E_2, I_2 as shown in Fig.3. Load on the beam is taken as a concentrated load 'P_2' Newton acting at a distance of 'ζ' meters from left end 'C'. Equation for its deflected neutral axis is given by:

$$v_2(x) = P_2\{L < x - \zeta >^3 \\ - (L - \zeta)x(x^2 - 2L\zeta + \zeta^2)\} / 6LE_2 I_2 \qquad (21)$$
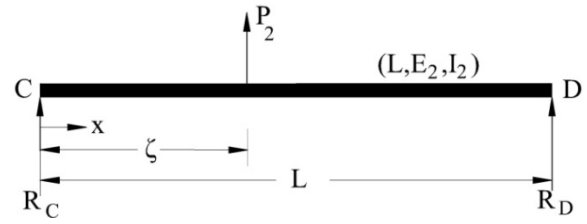

Figure 3. Standard Beam for Deflection (SBD): $TSB2$/ TEB_2

The beam, shown in Fig.(4), will be taken as standard beam for slope (SBS). It is defined as a simply supported beam 'CD' with an anticlockwise moment 'M_2' acting at a distance 'ζ' from the left end. The beam parameters are taken as E,I, L. SBS will be designated as TSB_2 / TEB_2 as the situation demands. This SBS can be used for getting slope of the deflected neutral axis of given beams. Equation of its deflected elastic curve is given by:
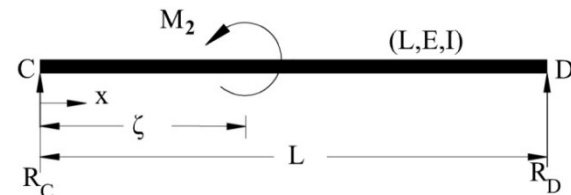

Figure 4. Standard Beam for Slope (SBS): TSB_2/ TEB_2

$$v_2(x) = M_2\{x(x^2 + 2L^2 + 3\zeta^2 - 6L\zeta) \\ - 3L < x - \zeta >^2\} / 6LEI \qquad (22)$$

4. Deflection Using PQW

In this section five illustrations are included to illustrate the use of PQW for axial bar, torsion rod and beams. In all these illustrations two systems are chosen as TES of each other and having same length. By opting for TES it becomes easy as one can use quasi work expressions which make internal force/ torque/ bending moment expressions redundant. As these expressions are not needed, one need not

have the knowledge of writing these expressions thereby making the process of obtaining deflections easier and faster. Two illustrations one each for axial bar and shaft, even though trivial, are included to impart clarity in understanding of concepts involved.

4.1. Application to Axial Bars

In this section equation for deformation of fixed - fixed bar given in Fig.(1a) is obtained by using the equation of deformation of a built-in bar given in Fig(1b). As it is easy to obtain quasi work, these bars will be chosen as TEAB having same parameters L, A and E.

Eqn.(12) with modified parameters yields:

$$W_{12} = \{P_2 / AE\}_2 \{[-R_B\zeta + P_1(a - < a - \zeta >)]\}_1 \quad (23)$$

$$\text{And} \quad W_{21} = \{P_2\}_2 \{u_1(\zeta)\}_1 \quad (24)$$

Using PQW in the form $W_{21} = W_{12}$ yields:

$$\{u_1(\zeta)\}_1 = \{[-R_B\zeta + P_1(a - < a - \zeta >)] / AE\}_1 \quad (25)$$

R_B is calculated by using the boundary condition that $u_1(L) = 0$ and comes out to be equal to $P_1 a / L$. Substituting this value of R_B in the above equation, the expression for axial deformation of the given fixed – fixed axial bar becomes:

$$\{u_1(\zeta)\}_1 = \{P_1[a(1 - \zeta / L) - < a - \zeta >] / AE\}_1 \quad (26)$$

As ζ is an arbitrary point in $TEAB_2$, it can as well be replaced by 'x' and if curly brackets are also dropped then Eqn. (26) takes conventional form.

4.2. Application to Rods

In this section PQW will be applied to a fixed – fixed shaft, shown in Fig.5, to obtain its equation of twist by making use of the equation of twist of the standard rod shown in Fig.2b.

The given shaft in Fig.5 carries a distributed torque of 't' N.m/m and has parameters G, J, L. Let T_A and T_B be the reaction torques in anticlockwise direction acting at left support 'A' and right support 'B', respectively. This shaft will be designated as TER_1.
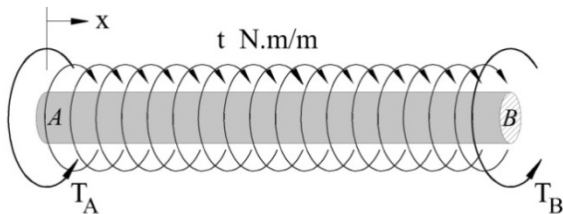
Figure 5. Fixed - Fixed Shaft: TER_1

Standard torsion rod shown in Fig.2b will be chosen to have same parameters G, J, L so that it is topologically equivalent to the given shaft and will be referred to as TER_2. Equation for twist as a function of 'x' for TER_2 is given by:

$$\{\theta_2(x)\}_2 = [T_2 x - T_2 < x - \zeta >] / GJ \quad (27)$$

W_{12} is calculated as follows:

$$W_{12} = \{T_A\}_1\{\theta(0)\}_2 + \int_0^L \{-t\}_1\{\theta(x)\}_2 + \{T_B\}_1\{\theta(L)\}_2$$

$$= (\{-T_2 t[x^2 - < x - \zeta >^2]_0^L\} / 2 + T_2 T_B \zeta) / GJ \quad (28)$$

$$= \{T_2\}_2\{\zeta[t(\zeta / 2 - L) + T_B)] / GJ\}_1$$

$$\text{And} \quad W_{21} = \{T_2\}_2\{\theta_1(\zeta)\}_1 \quad (29)$$

Using PQW in the form $W_{21} = W_{12}$ yields:

$$\{\theta_1(\zeta)\}_1 = \zeta[T_B + t(\zeta / 2 - L)] / GJ \quad (30)$$

T_B is obtained to be equal to $tL/2$ by applying the boundary condition $\theta(L) = 0$. Substituting this value in Eqn.(30), expression for twist of fixed - fixed shaft becomes:

$$\{\theta_1(\zeta)\}_1 = t\zeta(\zeta - L) / 2GJ \quad (31)$$

As ζ is an arbitrary point in TER_2 it is also an arbitrary point in the given problem (TER_1) which has same length as that of TER_2, hence, represents the same location as in (TER_1). Therefore, if ζ is replaced by 'x' and curly brackets are dropped, Eqn.(31) will take the conventional form.

4.3. Application to Beams

In this section PQW will be applied to beams. Three beam problems are chosen for this purpose. First and second problem is for obtaining the expression for deflection of a determinate and an indeterminate beam, respectively. Last one demonstrates the use of PQW for getting the expression for slope of a determinate beam.

Illustration – 3: Deflection of Determinate Beam

TEB_1 represents the given problem which in this illustration is chosen as a simply supported beam 'AB' of length 'L' with an overhang of length 'a' to the left of the support 'A' as shown in Fig.6. The loading on it is taken as an anticlockwise moment M_1 acting at the free end and a uniform load of intensity w_1 N/m between the supports. Let its parameters be E and I (second moment of area). It is required to find the equation for the deflected neutral axis between the supports. SBD (Fig.3) is chosen as TEB_2 having same parameters E, I and L.
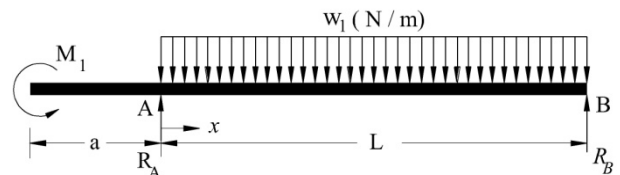
Figure 6. Beam with overhang: TEB_1

As the equation for neutral axis between the supports is required, the end moment M_1 is statically transferred to support 'A'. W_{12} is calculated as follows:

$$W_{12} = \{M_1\}_1\{v_2'(0)\}_2 + \int_0^L \{-w_1\}_1\{v_2(x)\}_2 dx$$

$$+ \{R_A\}_1\{v_2(0)\}_2 + \{R_B\}_1\{v_2(L)\}_2 \quad (32)$$

$$= -(P_2 / EI)[M_1(L - \zeta)(\zeta^2 - 2L\zeta) / 6L$$

$$+ w_1(L - \zeta)(L^2\zeta + L\zeta^2 - \zeta^3) / 24]$$

$$W_{21} = \{P_2\}_2\{v_1(\zeta)\}_1 \quad (33)$$

As the two beams were chosen to be TES of each other, deflection at any point ζ in TEB_1 is obtained by using PQW in the form $W_{21} = W_{12}$ and comes out to be:

$$\{v_1(\zeta)\}_1 = -(L - \zeta) [w_1(L^2\zeta + L\zeta^2 - \zeta^3)$$
$$+ 4M_1(\zeta^2 - 2L\zeta) / L] / 24EI \quad (34)$$

If $M_1 = w_1 L^2/12$ then Eqn.(34) simplifies to:

$$\{v_1(\zeta)\}_1 = -w_1(3\zeta^4 - 7L\zeta^3 + 3L^2\zeta^2 + L^3\zeta)/72EI \quad (35)$$

Replacing ζ by x and dropping suffix '1' as well as curly brackets in the above equation yields the equation for deflection in conventional form which is same as given in[18]

Illustration - 4: Deflection of Indeterminate Beam

In this illustration equation for the deflected neutral axis of a beam AB which has a roller support at the left end 'A' and is built-in at support 'B' is obtained. Beam has parameters L, E, I and carries an anticlockwise moment 'M' at its mid point as shown in Fig.7 wherein R_A, R_B and M_B are reactions from supports. This beam is designated as TEB_1. SBD (Fig.3) is chosen as TEB_2.

Quasi work W_{12} is given by:

$$
\begin{aligned}
W_{12} = P_2[M\{3L < L/2 - \zeta >^2 \\
- (L - \zeta)(3L^2/4 + \zeta^2 - 2L\zeta)\} \\
- M_B(L - \zeta)\zeta(L + \zeta)]/6EIL \quad \text{and}
\end{aligned}
\quad (36)
$$

$$W_{21} = P_2 v_1(\zeta)$$

Using $W_{21} = W_{12}$, $v_1(\xi)$ comes out to be equal to:

$$
\begin{aligned}
v_1(\zeta) = [M\{3L < L/2 - \zeta >^2 \\
- (L - \zeta)(3L^2/4 + \zeta^2 - 2L\zeta)\} \\
- M_B(L - \zeta)\zeta(L + \zeta)]/6EIL
\end{aligned}
\quad (37)
$$

Using boundary condition that slope at $\zeta = L$ is zero, yields value of M_B as M/8. With this value of M_B the equation for the deflection becomes:

$$
\begin{aligned}
v_1(\zeta) = M[24L < L/2 - \zeta >^2 \\
- 8(L - \zeta)(3L^2/4 + \zeta^2 - 2L\zeta) \\
- (L - \zeta)\zeta(L + \zeta)]/48EIL
\end{aligned}
\quad (38)
$$

This on simplification yields:

$$v_1(\zeta) = M[9\zeta^3/L - 3L\zeta + 24 < \zeta - L/2 >^2]/48EI \quad (39)$$

Replacing ζ by x and substituting 2L for L yields the same equation as given in[18]

Illustration – 5: Slope of Determinate Beam

Problem given in illustration - 3 is here taken up again for obtaining the equation of the slope of the neutral axis. TEB_2 is chosen as SBS (Fig.4) with equation for deflection given by Eqn.(22). Quasi work, W_{12}, is calculated as follows:

$$
\begin{aligned}
W_{12} = \{M_1\}_1\{v_2'(0)\}_2 + \int_0^L \{-w_1\}_1\{v(x)\}_2 \, dx \\
+ \{R_A\}_1\{v_2(0)\}_2 + \{R_B\}_1\{v_2(L)\}_2 \\
= M_2[12M_1(2L^2 + 3\zeta^2 - 6L\zeta)/L \\
- w_1(3L^3 - 18L\zeta^2 + 12\zeta^3)]/72EI \quad \text{and}
\end{aligned}
\quad (40)
$$

$$W_{12} = M_2 v_1'(\zeta)$$

Using $W_{21} = W_{12}$, expression for slope is given by:

$$
\begin{aligned}
v_1'(\zeta) = [12M_1(2L^2 + 3\zeta^2 - 6L\zeta)/L \\
- w_1(3L^3 - 18L\zeta^2 + 12\zeta^3)]/72EI
\end{aligned}
\quad (41)
$$

If M_1 is taken to be equal to $w_1L^2/12$, then the above equation reduces to:

$$v_1'(\zeta) = -w_1(12\zeta^3 - 21L\zeta^2 + 6L^2\zeta + L^3)/72EI \quad (42)$$

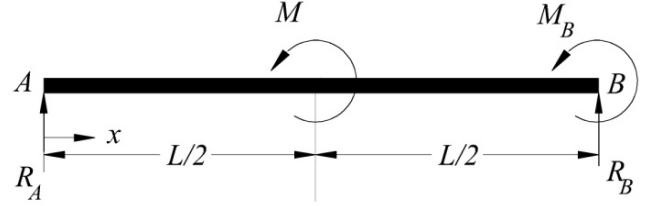As expected above equation is the derivative of Eqn.(35)

Figure 7.   Indeterminate Beam: TEB_1

5. Conclusions

1). PQW can be applied to a set of any two axial bars.

2). PQW is applicable to a pair of rods/ shafts so long as these have same radius of cross section.

3). Axial, torsion and beam standard elements are defined. With the help of the equation of deformed/ deflected elastic line of these predefined standard elements and PQW solution for deflection of other problems of the same class is obtained with ease.

4). Obtaining deflection/deformations at a point or equation of deflected/deformed elastic line of given problems using PQW is further simplified if the given problem and its corresponding standard element are chosen to be topologically equivalent. This is due to:

i) The knowledge of internal force (/torque /bending moment) is not required.

ii) For distributed loads one definite integral is to be evaluated whose integrant is the product of two a priory known expressions (viz. distributed load of the given problem and the equation of deflected elastic line of its standard element).

iii) For point loads it is simpler and easier as even the integration is not required. For this kind of loading,sum of multiplications of known loads of the given problem with known deflections/ deformations at the corresponding points in the chosen topologically similar system is all that is required.

5). As solutions of standard elements (for deflection/ deformation of its elastic line) are used to solve any other problem of the same class with different loads and different constraints, it is possible to develop an interactive graphic general purpose computer package for this purpose.

6). The concepts of virtual force and complementary energy do not exist in PQW as both the systems are real. Hence, no additional effort is needed for learning.

7). PQW is used for solving indeterminate problems with the help of the solution of their determinate topologically similar system.

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