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# Derivative Theorems of the Principle of Quasi Work

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**Abstract** Derivative theorems of Principle of Quasi Work, a powerful pseudo energy principle, are derived. These theorems, viz. Applied Load Theorem, Deflection Theorem and Unit Load Theorem are applicable to a pair of topologically similar structural systems. One more theorem referred to as Relative Deflection Theorem along with its two corollaries are also derived from this principle for facilitating truss analysis. Using these theorems, a new methodology for calculating nodal deflections of a truss from its internal member forces and vice versa is presented in this paper. This methodology is amazingly simple, easy, and fast. These theorems form the basis of present methodology. Thirteen nodal deflections of a four bay truss included in this paper were calculated by hand in less than fifteen minutes.

**Keywords** Continuum Mechanics, Energy Methods, Structural Analysis, Truss Deflection, PQW, TSS, TST, TET

## 1. Introduction

All the presently available structural analysis procedures e.g. finite element methods ([1 - 4]) variational principles ([5, 6]) and energy methods([7, 8]), are meant for a single structure and do not provide any connection between two structural systems.The **P**rinciple of Quasi Work (PQW) derived by [6] establishes a connection between Topologically Similar Systems (TSS) in the realm of structural mechanics. PQW thus fills the existing void in the domain of structural mechanics. Theorems based on PQWwhich are useful for discrete structural modelshave been derived by [9, 10].PQW was used advantageously by [9] and [11] for obtaining redundant reactions of beams. PQW has wide applicability as all the existing energy principles are derived as special cases of PQW by restricting TSS to Topologically Identical Systems (TIS), i.e. when two systems are identical with each other in every respect.

In this paper, four theorems and two corollaries are derived from PQW. Applied Load Theorem, Deflection Theorem and Unit Load Theorem are applicable to any two topologically similar structural systems. Applied Load Theorem is useful for obtaining loads acting on any given system by making use of the corresponding known deflections in its TSS. Similarly, Deflection Theorem and Unit Load Theorem are aptfor calculating deflections of a given structural system by using the known solution of its TSS. A given system can be determinate or indeterminate whereas its TSS is invariably determinate as its solution is used to solve the given problem. TSS of a given system canalso be indeterminate as long as its solution is known. For a class of problem (e.g. beams,plates and shells) one TSS whose solution is known is sufficient to solve other problems in that class with much easecompared to solving it by classical methods.

The last derived theorem, Relative Displacement Theorem (RDT), and its corollaries considerably simplify the calculation of nodal deflections from internal member forces of determinate or indeterminate trusses. Internal member forces can also be calculated form nodal deflections by RTD in a simplified way. Some illustrations are also included. In the following section a brief introduction to PQW is given.

## 2. Principleof Quasi Work

The Principle of quasi work was first derived by [6] and subsequently stated by [7] and [8]. According to this principle quasi work ($W_{mn}$) done by a self equilibrating system of external forces, $\{P\}_m$, of a structural system (say 'm') while undergoing the corresponding compatible displacements, $\{d\}_n$, of the other topologically similar system (say 'n') is equal to the quasi strain energy($U_{mn}$) due to internal forces, $\{F\}_m$, of the former system m while going through corresponding deformations, $\{\delta\}_n$, of the latter system n. In mathematical terms this can be stated as:

$$\{P\}_m^T \{d\}_n = \{F\}_m^T \{\delta\}_n \tag{1}$$

$$\text{OR} \qquad W_{mn} = U_{mn} \tag{2}$$

Where,

$U_{mn}$ = Quasi strain energy= $\{F\}_m^T \{\delta\}_n$;

$W_{mn}$ = Quasi work= $\{P\}_m^T \{d\}_n$;

$\{F\}_m$ = Internal member forces of the truss represented by $TSS_m$;

$\{P\}_m$= External nodal loads on the truss represented by $TSS_m$;

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$\{d\}_n$ = Displacement in $TSS_n$ corresponding to $\{P\}_m$;

$\{\delta\}_n$ = Deformation of the member in $TSS_n$ corresponding to $\{F\}_m$ in $TSS_m$;

$\{*\}^T$ = Transpose of $\{*\}$;

m, n = Subscripts representing different TSS. (m, n = 1, 2…., $m \neq n$).

This principle establishes a link between two systems (m and n) which should satisfy the following three conditions:

1. Thesystems should be topologically similar. TSS, as explained by [9], have the same number nodes and similar nodal interconnectivity. It is self evident when discrete structural models or even trusses or frames are involved, as in these cases nodes are well defined and hence the nodal interconnectivity is also obvious. In other cases of continuum structural elements it is not as obvious. All continuum structural elements can be considered as having infinite nodes (atoms or molecules) with well defined interconnectivity (forces between these). With this extension of definition of TSS, PQW can be applied to continuum elements like axial bars, torsion rods, beams, plates and other elements as well as to full fledged structures. Interestingly, what connects two nodes does not matter. In other words constitutive relations and material properties of the elements connecting different nodes have no place in the concept of TSS.

Detailed illustrations of TSS for discrete structural models, trusses and beams are given in [8 - 10].

2. Displacements and deformations of a TSS should be compatible within the system.

3. External forces acting on a TSS should be self equilibrating. Hence, all the reactions originating from the constraints form part of the external loads.

While applying this principle, attention has to be paid to the word 'corresponding' in the statement of PQW. For example take two axial bars (truss members) 'AB' and 'CD' of lengths '$L_1$' and '$L_2$'. As both the axial bars have infinite nodes, in these two axial bars end 'A' corresponds to end 'C' and end 'B' to end 'D'. In such cases, one has to resort to mapping in the two domains. For example, a point $0.64L_1$ from end 'A' in axial bar 'AB' will have its corresponding point at $0.64L_2$ from end 'C' in axial bar 'CD'.

# 3.Derivative Theorems of PQW

From PQW, theorems which are useful for calculating loads and deflections in topologically similar structural systems are derived hereunder. In PQW, the problem at hand represents one of the systems and its topologically similar system is exclusively the choice of the analyst. In these theorems, loads include moments and deflections include rotations; i.e., the terms 'loads' and 'deflections' are used in a generalized sense.

### 3.1. Load Theorem for Topologically Similar Structures

This theorem relates two TSS and can be used to find the load acting at a point in a system with the help of deflection at its corresponding point in another TSS.

STATEMENT: *"In a pair of Topologically Similar Systems (m and n); gradient of Quasi Strain Energy / Quasi Work with respect to deflection at any point in one of the systems (say n) is equal to load acting at the corresponding point in the other system (m)and is in the direction of the displacement."*

In mathematical form, this is stated as:

$$\frac{\partial U_{mn}}{\partial d_{jn}} = \frac{\partial W_{mn}}{\partial d_{jn}} = P_{jm} \qquad j = 1, 2, ...., N \tag{3}$$

**PROOF:** From the definition of PQW we have:

$$U_{mn} = W_{mn} = \{P\}_m^T \{d\}_n \tag{4}$$

The gradient of $U_{mn}$ (Eqn.4) with respect to displacement $d_{jn}$ yields Eqn.(3).

### 3.2. Deflection Theorem for Topologically Similar Structures

This theorem relates two TSS and is used to determine deflection at any desired point in one system with the help of load acting at the corresponding point in second system.

STATEMENT:*"In a pair of Topologically similar systems ( m and n); gradient of Quasi Strain Energy / Quasi Work with respect to a point load acting in one of the systems (say m) is equal to the deflection in the direction of the point load at the corresponding point in the other system (n)."*

$$\frac{\partial U_{mn}}{\partial P_{jm}} = \frac{\partial W_{mn}}{\partial P_{jm}} = d_{jn} \qquad j = 1, 2, ...., N \tag{5}$$

**PROOF:** The gradient of $U_{mn}$ (Eqn.4) with respect to force $P_{jm}$ yields Eqn. (5).

Eqns. (3 and 5) have forms similar to the familiar Castigliano's theorems except that these govern a pair of two topologically similar systems and the concept of complimentary energy does not exist in PQW. It shall be realized that when the pair of systems are restricted to be TIS (i.e., these are clones of each other), these equations produce the same results as those due to Castigliano's theorems; thereby, establishing the fact that the present theorems are more general forms of the statement of these classical theorems.

When in any two TSS structural stiffness (e.g., AE/ GJ/ EI) is the samethen these systems are designated by Topologically Equivalent Systems, (TES), and for these systems, Eqn.(5) becomes:

$$\frac{\partial W_{mn}}{\partial P_{jm}} = \frac{\partial U_{mn}}{\partial P_{jm}} = \frac{\partial U_{nm}}{\partial P_{jm}} = \frac{\partial W_{nm}}{\partial P_{jm}} = d_{jn} \quad j = 1, ..., N \tag{6}$$

### 3.3. Unit Load Theorem for TSS

Unit load theorem is used to obtain deflections in one system by applying a unit load in the other TSS.

STATEMENT: "*In a pair of Topologically similar systems (m and n); Quasi Strain Energy / Quasi Work with respect to a unit load in one of the systems (say m) is equal to the deflection in the direction of the unit load at the corresponding point in the other system (n).*"

In mathematical form, it is stated as:

$$\bar{U}_{mn} = \bar{W}_{mn} = d_{jn} \tag{7}$$

Where, $\bar{U}_{mn}$ and $\bar{W}_{mn}$ are $U_{mn}$ and $W_{mn}$ for a unit load applied in $TSS_m$.

**PROOF:** If in $TSS_m$ only one load acts at the location '*j*' and if this load is equal to unity then $\{P\}_m^T$ = $P_{jm}$ (= 1). In such a case, PQW takes the following form

$$\bar{U}_{mn} = \bar{W}_{mn} = \{P\}_m^T\{d\}_n = P_{jm}d_{jn} = d_{jn} \tag{8}$$

This completes the proof. For TES, Eqn.(7) takes the form

$$\bar{W}_{mn} = \bar{U}_{mn} = \bar{U}_{nm} = \bar{W}_{nm} = d_{jn} \tag{9}$$

# 4. Relative Deflection Theorem for Trusses

A theorem for calculating relative deflection of two nodes with respect to each other connected by a truss member is stated and proved. This theorem significantly simplifies the calculations needed for obtaining nodal deflections and member forces of any determinate or indeterminate truss.

*STATEMENT: "In a given truss the relative displacement '$d_{i/j}$' of any two nodes 'i' and 'j' is equal to the deformation '$\delta_k$' of the truss member 'k' connecting these two nodes and is along the axis of the truss member."*

In mathematical form it can be written as:

$$d_{i/j} = d_i - d_j = \delta_k \ (= F_k\ L_k\ /\ A_kE_k) \tag{10}$$

**PROOF:** Take a truss having 'N' nodes with 'M' axial members. For any given self equilibrating loading, let $F_n$ be the internal force in these truss members. This given truss is denoted by $TST_1$. A topologically similar truss, $TST_2$, is derived from the given truss by assigning zero stiffness to all truss members except one member 'k' which connects nodes '*i*' and '*j*'. Let this member carry a self equilibrating tensile load '$P_2$'. Load acting at node '*i*' will be taken as positive and hence load at node '*j*' will be negative. The force $\{F_k\}_2$ in this member is equal to $P_2$.

Quasi work $W_{21} = \{P_2\}_2\ \{d_i\}_1 - \{P_2\}_2\ \{d_j\}_1$

$$= \{P_2\}_2\ (\{d_i\}_1 - \{d_j\}_1) \tag{11}$$

Quasi strain energy $U_{12} = \{P_2\}_2\{\delta_k\}_1$ (12)

PQW ($W_{21} = U_{21}$) yields Eqn.(10) after dropping curly brackets along with suffix '1'.

### 4.1. Corollary - 1:

"*The relative displacement of the last node with respect to the first node in collinearly connected 'n' truss members in series is equal to the sum of the individual deformation of these 'n' truss members and is along their axes.*"

In mathematical form it can be stated as:

$$d_{(n+1)/1} = d_{(n+1)} - d_1 = \sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \frac{F_j L_j}{A_j E_j} \tag{13}$$

**PROOF:** Relative displacement between nodes of different elements is given by:

$$d_{(n+1)/n} = d_{n+1} - d_n = \delta_n = F_nL_n\ /A_n\ E_n \tag{14}$$

$$d_{n/(n-1)} = d_n - d_{n-1} = \delta_{n-1} = F_{n-1}L_{n-1}/A_{n-1}E_{n-1} \tag{15}$$

$$d_{3/2} = d_3 - d_2 = \delta_2 = F_2L_2/A_2E_2 \tag{16}$$

$$d_{2/1} = d_2 - d_1 = \delta_1 = F_1L_1/A_1E_1 \tag{17}$$

As these are connected in series with their axes collinear, the sum of all these equation yields Eqn.(13).

### 4.2. Corollary - 2:

"*In collinearly connected 'n' truss members of a truss whose first and last nodes are constrained, the deformation (and thus internal force) of at least one truss member is opposite in nature to the deformation of other 'n-1' members.*"

**PROOF:** From corollary - 1, displacement of the last node with respect to the first node is zero as both these nodes are constrained. Hence,

$$d_{(n+1)/1} = d_{(n+1)} - d_1 = 0 = \sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \frac{F_j L_j}{A_j E_j} \tag{18}$$

The last two sums in Eqn.(18) can not be zero unless at least one member deformation in the first sum and at least one member force in the second sum is of opposite sign.

Relative displacement theorem and its corollaries simplify the procedure for calculating nodal deflections and member forces in any given truss. To illustrate the use of the given theorem and its corollary three illustrations shown in Figs. 1-3 follow.

# 5. Nodal Deflections of Trusses

In this section, three examples are chosen to illustrate the procedure of obtaining nodal deflections of trusses. These illustrations are depicted in Fig.1 to Fig.3. Fig.1 shows a three bay truss with one internal and one external indeterminacy. Fig.2 depicts a four bay truss with two degree internal indeterminacy and one-degree of external indeterminacy. Fig.3 shows a two storey truss with two-degree internal indeterminacy. In what follows, nodal deflections (*u* and *v*) of the given truss ($TSS_1$) are represented by dropping suffix '1'.

**Fig. 1: Three bay truss**

Fig.1a depicts a three bay truss $A_1\ B_1...F_1$ having one internal and one external degree of indeterminacy is solved for nodal deflections. Young's modulus of elasticity,'E', for all members is 205 kN/mm². This given truss isdesignated by $TST_1$. Member forces are obtained by any convenient conventional method after which nodal deflections are obtained by using RDT and its corollaries in the following manner. Let

$u_*$ = Horizontal deflection of truss joint '*' (* represents: A, B, C, .....).

$v_*$= Vertical deflection of truss joint '*' (* represents: A, B, C, .....).

From the given support conditions $u_a = v_a = v_c = v_d = 0$. Now utilizing RTD and its corollaries, horizontal deflection of nodes ‘$B_1$’, ‘$C_1$’, ‘$D_1$’ and vertical deflection of node ‘$F_1$’ are calculated as follows:

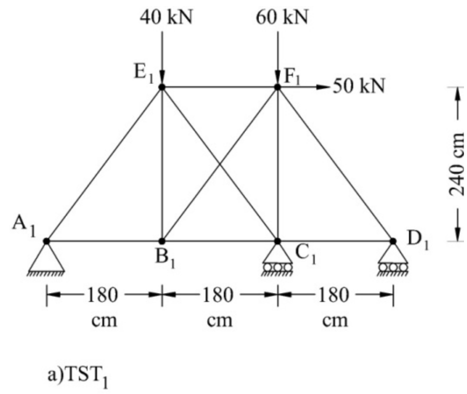
$u_b - u_a = \delta_{ab} = 0.2179$ mm (from Table1, Column 6) which yields $u_b = 0.2179$ mm. Similarly, $u_c = \delta_{ab} + \delta_{bc} = 0.3912$ mm, $u_d = u_c + \delta_{cd} = 0.4757$mm and $v_f = \delta_{cf} = -0.2335$ mm.

In order to calculate other nodal deflections (i.e.,$v_b$, $u_e$, $v_e$, and $u_f$) by utilizing RDT, vertical deflection of node ‘E’ or ‘B’and horizontal deflection of node ‘E’ or ‘F’ should be known. Here let us choose to calculate deflections (u, v) of node ‘E’ by using deflection theorem for which $TST_2$ has to be defined.

Fig.1b shows $TST_2$which is derived from the given truss by assigning zero stiffness (AE = 0) to all truss members connected to nodes $D_1$ and $F_1$. Thus $TST_2$ is a determinate truss $A_2B_2C_2E_2$ having only five members. Loads on $TST_2$ are applied at those nodes whose deflection is to be calculated. Hence, at node ‘$E_2$’ a horizontal load $P_1$kN and avertical load $P_2$kN are applied as the deflection in these direction is to be calculated. Internal forces in truss members of $TST_2$ are given in column 7 of Table 1. Column 8 of the table gives contribution of truss members to $U_{21}$ and the sum of this column given in the last row is equal to $U_{21}$. By using the deflection theorem the deflections of node ‘$E_1$’ are $u_e = \partial U_{21}/\partial P_1 = 0.3349$ mm and $v_e = \partial U_{21}/\partial P_2 = -0.2474$ mm.

Other two nodal deflections ($v_b$ and $u_f$) are once again obtained by relative deflection theorem. $v_e - v_b = \delta_{eb} = -0.0661$ which yields $v_b = -0.1813$ mm. Similarly, $u_f = 0.4252$ mm. All these nodal deflections (in mm) are summarized below:

$u_a = 0$, $v_a = 0$; $u_b = 0.218$, $v_b = -0.181$; $u_c = 0.391$, $v_c = 0$;$u_d = 0.476$, $v_d = 0$; $u_e = 0.335$, $v_e = -0.247$ and $u_f = 0.425$, $v_f = -0.234$. These values are the same as obtained by conventional methods.



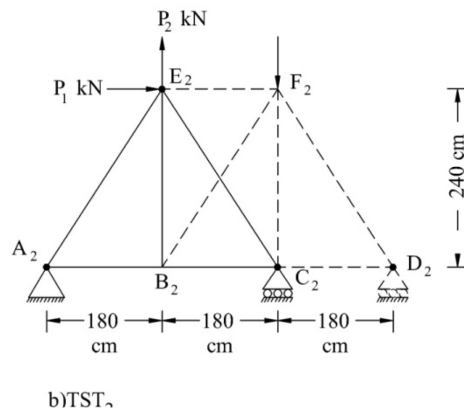




**Figure 1.** Three bay Truss

Thus, once member forces of a truss are known, the present method makes it possible to calculate all the nodal deflections easily and quickly without any need of a computer. In fact, all the above nodal deflections were obtained in about ten minutes by hand calculation with the help of abusiness pocket calculator.

**Table 1.** Calculations for theThree Bay Truss Shown in Fig.1a

| Given Truss (Fig. 1a):: $TST_1$ | | | | | | Chosen Truss : (Fig.1b) :: $TST_2$ | |
|---|---|---|---|---|---|---|---|
| MEMBER No.‘j’ | Nodes | $A_j$ $cm^2$ | $L_j$ cm | Internal Force $\{F_j\}_1$:: kN | Deformation $\{\delta_j\}_1=\{F_jL_j/A_jE\}_1$ | Internal Force: $\{F_j\}_2$kN | Individual contribution to $U_{21}$ :: $\alpha_j = \{\delta_j\}_1 \{F_j\}_2$ |
| 1 | $A_nB_n$ | 20 | 180 | 49.6299 | 0.2179 | $0.5 P_1 - 0.375 P_2$ | $0.1089P_1 - 0.0817P_2$ |
| 2 | $B_nC_n$ | 20 | 180 | 39.4628 | 0.1733 | $0.5 P_1 - 0.375 P_2$ | $0.0866P_1 - 0.0650 P_2$ |
| 6 | $C_nD_n$ | 20 | 180 | 19.2598 | 0.0845 | 0 | 0 |
| 3 | $E_nF_n$ | 20 | 180 | 20.5731 | 0.0903 | 0 | 0 |
| 4 | $B_nE_n$ | 24 | 240 | -13.5561 | -0.0661 | 0 | 0 |
| 5 | $C_nF_n$ | 24 | 240 | -47.8764 | -0.2335 | 0 | 0 |
| 7 | $A_nE_n$ | 30 | 300 | 61.6855 | 0.0030 | $5P_1/6 + 0.625 P_2$ | $0.0025 P1 + 0.0019 P_2$ |
| 8 | $B_nF_n$ | 30 | 300 | 19.9451 | 0.0827 | 0 | 0 |
| 9 | $D_nF_n$ | 30 | 300 | -32.0996 | -0.1566 | 0 | 0 |
| 10 | $C_nE_n$ | 30 | 300 | -33.6717 | -0.1643 | $-5P_1/6+0.625P_2$ | $0.1369P_1 - 0.1026 P_2$ |
| | | | | | | $U_{21}$ = SUM of $\alpha_j$ (for j = 1...10) = | $0.3349P_1$- 0 $.2474P_2$ |

**Fig2: Four bay truss**

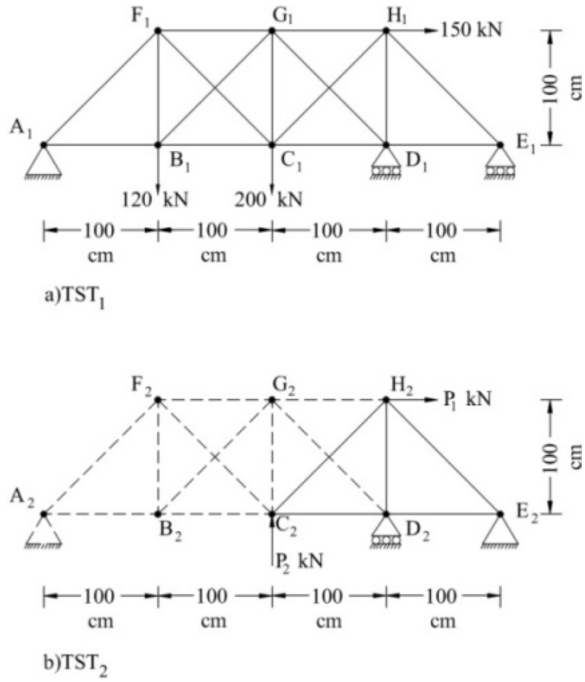



**Figure 2.** Four Bay Truss

Fig.2a depicts a four bay truss '$A_1B_1...H_1$' with two- degree internal and one- degree external indeterminacyis taken for illustrating the use of PQW and relative deflection theorem. E for all members is E = 205kN/mm$^2$.From the given support conditions $u_a = v_a = v_d = v_e = 0$. Horizontal deflection of nodes '$B_1$', '$C_1$', '$D_1$', '$E_1$' and vertical deflection of node '$H_1$' are obtained as follows.From relative deflection theorem, $u_b$ – ua = $\delta_{ab}$ = 0.2344 (column 6 of Table 2) which yields $u_b$ = 0.2344 mm. Utilizing corollary of relative deflection theorem, $u_c - u_a = \delta_{ab} + \delta_{bc}$ = 0.4193 mm, which gives $u_c$ = 0.4193 mm. Similarly, $u_d$ = 0.5046 mm, $u_e$ = 0.4859 mm and $v_h$ = -0.1388 mm. For calculating other deflections in the above manner, horizontal deflection of anyone among '$F_1$','$G_1$' and '$H_1$' nodes and vertical deflectionof one among '$B_1$','$C_1$','$F_1$', and '$G_1$' must be known. Let us choose to calculate $u_h$ and $v_c$. These will be obtained by using PQW for which $TST_2$ along with loading and supports are chosen as shown in Fig. 2.

All members connected to nodes $A_2$, $B_2$ and $G_2$ are assigned zero stiffness and the roller support at $E_1$ in $TST_1$ is replaced by a pinned support. Internal member forces in $TST_2$ are given in column7 of Table 2. Last column ofTable2 gives contribution of truss members to $U_{21}$ and the sum of this column (in the last row) gives $U_{21}$.

By using deflection theorem $v_c = \partial U_{21}/\partial P_2$ = - 0.6038 mm. Node 'E' has a horizontal reaction equal to $-P_1$, hence$\partial U_{21}/\partial P_1 = u_h - u_e$ = -0.1917 mm which yields $u_h$ = 0.2942 mm. Now relative displacement theorem yields $u_g$ = 0.2493, $u_f$ = 0.3579 mm and $v_g$ = -0.5494 mm.

**Table 2.** Nodal Deflection Calculations for the Four Bay Truss Shown in Fig.2a

| Given Truss (Fig. 2a):: $TST_1$ | | | | | | Chosen Truss :: (Fig.2b) :: $TST_2$ | |
|---|---|---|---|---|---|---|---|
| MEMBER | | $A_j$ cm$^2$ | $L_j$ cm | Internal Force $\{F_j\}_1$: kN | Deformation $\{\delta_j\}_1 = \{F_jL_j / A_jE\}_1$ | Internal Force: $\{F_j\}_2$kN | Individual contribution to $U_{21}$ :: $\alpha_j = \{\delta_j\}_1 \{F_j\}_2$ |
| No.'j' | Nodes | | | | | | |
| 1 | $A_nB_n$ | 50 | 100 | 240.2813 | 0.2344 | 0 | 0 |
| 2 | $B_nC_n$ | 50 | 100 | 199.5042 | 0.1849 | 0 | 0 |
| 3 | $C_nD_n$ | 50 | 100 | 87.4107 | 0.0853 | $P_2$ | 0.0853 $P_2$ |
| 4 | $D_n E_n$ | 50 | 100 | -19.158 | -0.0189 | $P_2$ | -0.0187 $P_2$ |
| 5 | $F_nG_n$ | 50 | 100 | -111.3385 | -0.1086 | 0 | 0 |
| 6 | $G_nH_n$ | 50 | 100 | 46.0073 | 0.0449 | 0 | 0 |
| 7 | $B_nF_n$ | 50 | 100 | 69.2229 | 0.0675 | 0 | 0 |
| 8 | $C_nG_n$ | 50 | 100 | 55.7916 | 0.0544 | 0 | 0 |
| 9 | $D_nH_n$ | 50 | 100 | -142.3093 | -0.1388 | $P_1$+ 2 $P_2$ | -0.1388$P_1$ - 0.2777 $P_2$ |
| 10 | $A_nF_n$ | 50 | 141.42 | -127.6762 | -0.1762 | 0 | 0 |
| 11 | $E_n$ $H_n$ | 50 | 141.42 | 27.0934 | 0.0374 | -1.4142 ( $P_1$+$P_2$) | - 0.0529$P_1$ - 0.0529$P_2$ |
| 12 | $B_nG_n$ | 50 | 141.42 | 71.8097 | 0.0991 | 0 | 0 |
| 13 | $D_nG_n$ | 50 | 141.42 | -150.7109 | -0.2079 | 0 | 0 |
| 14 | $C_nF_n$ | 50 | 141.42 | -29.7802 | -0.0411 | 0 | 0 |
| 15 | $C_nH_n$ | 50 | 141.42 | 174.1621 | 0.2403 | -1.4142 $P_2$ | -0.3398$P_1$ |
| $U_{21}$ = SUM of $\alpha_j$ (for j = 1...10) = | | | | | | | - 0.1917$P_1$- 0.6038$P_2$ |

The last two unknown deflections, $v_f$ and $v_b$, are calculated either using PQW (for which another TSS which includes node 'F' or 'B' is to be chosen) or by observing that $u_f$andthe displacement of node 'F' along AF ($\delta_{af}$) are already known. From these data one can obtain $v_f$ as follows. Let $u_{1f}$ and $v_{1f}$ be the displacements of node 'F' along and perpendicularto the axis of member 'AF'.In that case, $u_f = u_{1f}$ Cos $45^0$ - $v_{1f}$ Sin $45^0$ and $v_f = u_{1f}$ Sin $45^0$ + $v_{1f}$ Cos $45^0$. These two equations with $u_{1f} = \delta_{af}$=-0.1762 and $u_f$ = 0.3579yield $v_f$ = -0.6071 mm and $v_{1f}$ = -0.6823.Again, using the relative displacement theorem, $v_f - v_b = \delta_{bf}$ which gives $v_b$ = -0.6746 mm. All these nodal deflections (in mm) are summarized below:

$u_a = 0$, $v_a = 0$; $u_b = 0.234$, $v_b = -0.675$; $u_c = 0.419$, $v_c = -0.604$; $u_d = 0.505$, $v_d = 0$; $u_e = 0.486$, $v_e = 0$; $u_f = 0.358$, $v_f = -0.607$; $u_g = 0.249$, $v_g = -0.549$ and $u_h = 0.294$, $v_h = -0.139$. These values are the same as obtained by conventional methods.

**Fig.3: Two storey truss**

In this illustration, Fig. 3a, a two storey truss '$A_1B_1...F_1$' having two degree internal indeterminacy is solved for nodal deflections. It is pinned at joint '$A_1$' and has a roller support at joint '$B_1$'.

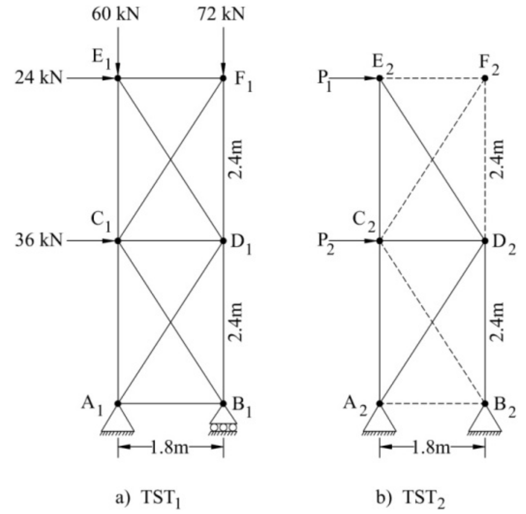



**Figure 3.** Two Storey Truss

**Table 3.** Nodal Deflection Calculations for theTwo Storey Truss Shown in Fig.3a

| Given Truss (Fig. 3a):: $TST_1$ | | | | | | Chosen Truss :: (Fig.3b) :: $TST_2$ | |
|---|---|---|---|---|---|---|---|
| MEMBER No.'j' | Nodes | $A_j$ $cm^2$ | $L_j$ cm | Internal Force $\{F_j\}_1$::kN | Deformation $\{\delta_j\}_1 = \{F_jL_j / A_jE\}_1$ | Internal Force: $\{F_j\}_2$kN | Individual contribution to $U_{21}$ : $\alpha_j = \{\delta_j\}_1 \{F_j\}_2$ |
| 1 | $A_nB_n$ | 10 | 180 | 41.686 | 0.3752 | 0 | 0 |
| 2 | $C_nD_n$ | 10 | 180 | 11.877 | 0.107 | - $P_2$ | -0.10689 $P_2$ |
| 6 | $E_n F_n$ | 10 | 180 | 6.1911 | 0.0557 | 0 | 0 |
| 3 | $A_n C_n$ | 20 | 240 | 27.582 | 0.1655 | 1.3333 $P_1$ | 0.22066$P_1$ |
| 4 | $C_nE_n$ | 20 | 240 | -19.745 | -0.1185 | 1.3333 $P_1$ | -0.158$P_1$ |
| 5 | $B_nD_n$ | 20 | 240 | -128.42 | -0.7705 | -2.667 $P_1$ -1.333 $P_2$ | 2.05472$P_1$+1.02736$P_2$ |
| 7 | $D_nF_n$ | 20 | 240 | -63.745 | -0.3825 | 0 | 0 |
| 8 | $A_nD_n$ | 30 | 300 | 30.523 | 0.1526 | 1.667$P_1$ + 1.667$P_2$ | 0.25436$P_1$+0.254358$P_2$ |
| 9 | $D_nE_n$ | 30 | 300 | -50.318 | -0.2516 | -1.667 $P_1$ | 0.41932$P_1$ |
| 10 | $B_nC_n$ | 30 | 300 | -69.477 | -0.3474 | 0 | 0 |
| 11 | $C_nF_n$ | 30 | 300 | -10.319 | -0.0516 | 0 | 0 |
| | | | | | $U_{21}$ = SUM of $\alpha_j$ (for j = 1...10) = | 2.79109$P_1$ + 1.174825$P_2$ | |

**Table4.** Member Force calculations for truss shown in Fig.3a

| Deflection of the Nodes of the given Truss | | | | | | |
|---|---|---|---|---|---|---|
| Nodes | $A_1$ | $B_1$ | $C_1$ | $D_1$ | $E_1$ | $F_1$ |
| Hor. Def. $u_i$ | 0.0 | 0.37517 | 1.1748 | 1.2817 | 2.7911 | 2.8468 |
| Vert. Def. $v_i$ | 0.0 | 0.0 | 0.16549 | -0.77052 | 0.04702 | -1.1530 |

| Member Data and Internal Force Calculations for Truss Members | | | | | | |
|---|---|---|---|---|---|---|
| No. 'j' | Nodes | Length ($L_j$) mm | Area ($A_j$) $mm^2$ | Deformation By Deflection Theorem | $\delta_j$ (mm) | $F_j$ = A E $\delta_j$ / L(kN) |
| 1 | $A_1$ $B_1$ | 1800 | 1000 | $u_b - u_a$ | 0.3752 | 41.686 |
| 2 | $C_1$ $D_1$ | 1800 | 1000 | $u_d - u_c$ | 0.1067 | 11.877 |
| 3 | $E_1$ $F_1$ | 1800 | 1000 | $u_f - u_e$ | 0.0557 | 6.1911 |
| 4 | $A_1$ $C_1$ | 2400 | 2000 | $v_c - v_a$ | 0.1655 | 27.582 |
| 5 | $C_1$ $E_1$ | 2400 | 2000 | $v_e - v_c$ | -0.1185 | -19.745 |
| 6 | $B_1$ $D_1$ | 2400 | 2000 | $v_d - v_b$ | -0.7705 | -128.42 |
| 7 | $D_1$ $F_1$ | 2400 | 2000 | $v_f - v_d$ | -0.3825 | -63.745 |
| 8 | $A_1$ $D_1$ | 3000 | 3000 | $3/5\ u_d + 4/5\ v_d$ | 0.1526 | 30.523 |
| 9 | $D_1$ $E_1$ | 3000 | 3000 | $(-3(u_e - u_d) + 4*(v_e - v_d)) / 5$ | -0.2516 | -50.318 |
| 10 | $B_1$ $C_1$ | 3000 | 3000 | $(-3(u_c - u_b) + 4*(v_c - v_b)) / 5$ | -0.3474 | -69.477 |
| 11 | $C_1$ $F_1$ | 3000 | 3000 | $(3(u_f - u_c) + 4*(v_f - v_c)) / 5$ | -0.0516 | -10.319 |

E for all members is E = 200kN/mm$^2$. From support conditions $u_a = v_a = v_b = 0$. By using relative displacement theorem and its corollary we have:

$v_c = \delta_{ac}$ =0.165492 mm; $v_e = \delta_{ac} + \delta_{ce}$ = 0.047022 mm; $v_d = \delta_{bd}$ = -0.77052 mm; $v_f = v_d + \delta_{df}$ = -1.15299 mm and $u_b$ = 0.375174 mm.

In order to calculate horizontal deflection of nodes '$C_1$', '$D_1$', '$E_1$' and '$F_1$' by using relative displacement theorem, we need to know the horizontal displacements of one node each from the pair 'C', 'D' and 'E', 'F'. Let us chose to calculate the deflection of nodes 'C' and 'E'. This can be easily calculated by resorting to PQW for which $TST_2$ is to be chosen which should include the nodes 'C' and 'E' and is a stable truss under the chosen loading conditions. Fig. 3b showsone such $TST_2$.

Loads '$P_1$' and '$P_2$' are applied at the nodes and in the direction in which the deflection is to be calculated. Member forces in $TST_2$ are given in column 7 of Table 3. In column 8 are tabulated the contribution from individual members to $U_{21}$ in terms of $P_1$ and $P_2$. The last row gives the value of $U_{21}$, which is the sum of all these individual contributions. The partial derivative of $U_{21}$ with respect to $P_1$ and $P_2$ yields the values of $u_e$ = 2.79109 mm and $u_c$ = 1.174825 mm, respectively. Now by using RDT one can easily calculate $u_d$ = 1.281718 mm and $u_f$ = 2.8468099 mm. Deflection of different nodes in mm is summarized in second and third row of Table 4 and these values are same as obtained by conventional methods.

Thus one can observe that obtaining displacement of joints once the member forces are known is very easy compared to any conventional method.

## 6. Member Forces of Trusses

In this section member forces will be calculated from the given nodal deflections. The procedure adopted is simple and fast. All the calculations can easily be carried out using a pocket calculator. All this is possible because of the relative deflection theorem.

For illustrating the procedure the example in Fig.3 is again used. Here, starting point is the nodal displacements given in 2$^{nd}$ and 3$^{rd}$ row of Table 4. From these deflections internal forces in the members are calculated. In column 5 of Table 4 are given relations for member deformations got from RTD and column 6 gives their numerical values. In the last column member forces are tabulated, which are the same as given in Table 3 column 5.

## 7. Conclusions

1. PQW is successfully applied to a set of any two axial bars (Truss members).

2. Three theorems are derived from PQW. Deflection theorem and unit load theorem are useful for obtaining deflection in topologically similar structures. Their use has been illustrated for trusses.

3. Relative deflection theorem and its corollaries are derived from PQW for obtaining nodal deflections of trusses. These theorems greatly simplify the procedure for obtaining nodal deflection from given member forces.

4. In case of trusses, topologically similar trusses for a given problem can be chosen in such a way so as to minimise effort needed to solve the given problem. The chosen truss is derived from the given truss by assigned zero stiffness (AE) to some of the truss members in such a way so that the derived truss is determinate and remains stable under the chosen loading.

5. The concepts of virtual force and complementary energy do not exist in PQW as both the systems are real. Hence, no additional effort is needed for learning this method.

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